LU decomposition (Gauss Elimⁿ in disguise)

Solving $A\vec{x} = \vec{b}$

Let us say we are able to factorize $A \equiv LU$ (when (Gauss Elimⁿ) row-reduction can be performed w/o row-interchange)

s.t. $L = \begin{pmatrix} 1 & 0 & \cdots & & 0 \\ m_{21} & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & 0 & \vdots \\ & & & 1 & \\ m_{n1} & m_{n2} & \cdots & m_{n,n-1} & 1 \end{pmatrix}$ ← Lower Δ matrix

$U = \begin{pmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \cdots & a_{1n}^{(1)} \\ 0 & a_{22}^{(2)} & \cdots & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & a_{nn}^{(n)} \end{pmatrix}$ ← upper Δ matrix

this matrix is ref(A)

1st solve $LU\vec{x} = \vec{b}$
$\Rightarrow L\vec{y} = \vec{b}$ — (i)

then solve $U\vec{x} = \vec{y}$ — (ii)

final soln :- $\vec{x}$.

We will observe the mechanics of this w/ the help of an example.

Q1) Solve

$$\begin{aligned} x_1 + x_2 + 0x_3 + 3x_4 &= 4 \\ 2x_1 + x_2 - x_3 + x_4 &= 1 \\ 3x_1 - x_2 - x_3 + 2x_4 &= -3 \\ -x_1 + 2x_2 + 3x_3 - x_4 &= 4 \end{aligned}$$

We will reduce A to ref!

Factorize A (not $\hat{A}$!)

$$A = \begin{pmatrix} 1 & 1 & 0 & 3 \\ 2 & 1 & -1 & 1 \\ 3 & -1 & -1 & 2 \\ -1 & 2 & 3 & -1 \end{pmatrix} \xrightarrow{R_2 - \underset{m_{21}}{(2)}R_1 \to R_2} \begin{pmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 3 & -1 & -1 & 2 \\ -1 & 2 & 3 & -1 \end{pmatrix}$$

$$\xrightarrow{R_3 - \underset{m_{31}}{(3)}R_1 \to R_3} \begin{pmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & -4 & -1 & -7 \\ -1 & 2 & 3 & -1 \end{pmatrix} \xrightarrow{R_4 - \underset{m_{41}}{(-1)}R_1 \to R_4} \begin{pmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & -4 & -1 & -7 \\ 0 & 3 & 3 & 2 \end{pmatrix}$$

$$\xrightarrow{R_3 - \underset{m_{32}}{(4)}R_2 \to R_3} \begin{pmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & 3 & 13 \\ 0 & 3 & 3 & 2 \end{pmatrix} \xrightarrow{R_4 - \underset{m_{42}}{(-3)}R_2 \to R_4} \begin{pmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & 3 & 13 \\ 0 & 0 & 0 & -13 \end{pmatrix}$$

$m_{43} = 0$ b/c $R_4 - (0)R_3 \to R_4$ to obtain ref.

Some texts require the ref where the leading non-zero entry (pivot) = 1; other texts require zeros here → here we will not require the ref in which the pivots are 1.

Here, read off entries of U from the ref(A)

$$U = \begin{pmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & 3 & 13 \\ 0 & 0 & 0 & -13 \end{pmatrix}$$

read off entries of L by considering the circled multipliers of the successive row-transformations

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{pmatrix}$$

m_{21}, m_{31}, m_{41}, m_{32}, m_{42}, m_{43}

It can be easily verified that $A = LU$

Let us first solve $L\vec{y} = \vec{b}$ ——— (1)

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ -3 \\ 4 \end{pmatrix}$$

fwd sub.

$\Rightarrow y_1 = 4$ ——— (i)

$2y_1 + y_2 = 1 \Rightarrow y_2 = 1 - 2\times 4$

$y_2 = -7$ ——— (ii)

$3y_1 + 4y_2 + y_3 = -3$

$\Rightarrow 12 - 28 + y_3 = -3$

$\Rightarrow y_3 = -3 + 16 = 13$

$y_3 = 13$ ——— (iii)

$-y_1 - 3y_2 + y_4 = 4$

$\Rightarrow -4 + 21 + y_4 = 4$

$\Rightarrow y_4 = -13$ ——— (iv)

Now solve.

$U\vec{x} = \vec{y}$ ——— (11)

$$\Rightarrow \begin{pmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & 3 & 13 \\ 0 & 0 & 0 & -13 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 4 \\ -7 \\ 13 \\ -13 \end{pmatrix}$$

Bkd sub.

$x_4 = 1$ ——— (I)

$3x_3 + 13x_4 = 13$

$\Rightarrow x_3 = 0$ — (II)

$-x_2 - 5x_4 = -7$

$\Rightarrow x_2 = -5 + 7 = 2$

$x_2 = 2$ — (III)

$x_1 + x_2 + 3x_4 = 4$

$\Rightarrow x_1 = -1$ — IV

$\therefore \vec{x} = \begin{pmatrix} -1 & 2 & 0 & 1 \end{pmatrix}$

Sub in original eqn. & verify!

Q2) this is Q2 from last lecture.

$$x_1 + 4x_2 + 2x_3 = -2$$
$$-2x_1 - 8x_2 + 3x_3 = 32$$
$$x_2 + x_3 = 1$$

$$A = \begin{pmatrix} 1 & 4 & 2 \\ -2 & -8 & 3 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\Big\downarrow R_2 - (-2)R_1 \rightarrow R_2$$

$$\begin{pmatrix} 1 & 4 & 2 \\ 0 & 0 & 7 \\ 0 & 1 & 1 \end{pmatrix}$$

Clearly it will not be possible to perform row-red^n w/o row swapping; So let's try to solve the equivalent eq's.

$$x_1 + 4x_2 + 2x_3 = -2$$
$$x_2 + x_3 = 1$$
$$-2x_1 - 8x_2 + 3x_3 = 32$$

$$A = \begin{pmatrix} 1 & 4 & 2 \\ 0 & 1 & 1 \\ -2 & -8 & 3 \end{pmatrix}$$ * No transf^n was req/d for R_2 $\therefore m_{21} = 0$

$$\Big\downarrow R_3 - (-2)R_1 \rightarrow R_3$$ $m_{31} = -2$

$$U = \begin{pmatrix} 1 & 4 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 7 \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$ Indeed $A = LU$

1^{st} solve $\quad L\vec{y} = \vec{b}$ —— (i)

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 32 \end{pmatrix}$$

fwd sub $\Rightarrow$ $y_1 = -2$

$y_2 = 1$

$-2y_1 + y_3 = 32$

$\Rightarrow y_3 = 32 + 2\times(-2)$

$= 28$

then solve

$$U\vec{x} = \vec{y} \quad \text{—— (ii)}$$

$$\Rightarrow \begin{pmatrix} 1 & 4 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 28 \end{pmatrix}$$

bwd sub $\Rightarrow$ $7x_3 = 28 \Rightarrow x_3 = 4$

$x_2 + x_3 = 1 \Rightarrow x_2 = -3$

$x_1 + 4x_2 + 2x_3 = -2$

$\Rightarrow x_1 + (-12) + 8 = -2$

$\Rightarrow x_1 = 2$

$$\therefore \vec{x} = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}$$

this is exactly the answer we had obtained!

(1) In LU decomposition computational complexity reduces from $\underbrace{O(n^3/3)}_{\text{Gauss-Elim}^n}$ to $O(2n^2)$

(2) Uniqueness & existence

Generally speaking, the LU decomp^n is not unique (& may not exist)

If A is symmetric & +ve def (Hermitian) then $U = (L^{\dagger})^* = L^*$

& we have $A = LL^*$ known as the Cholesky decomp^n (always exist & unique)