Lecture (6) 24/1/2019

(6.1) Examples and applications of CR eqns.

Q.1) Choose the constant "a" so that the f^n
$$u(x,y) = x^3 + axy^2$$ is harmonic.
Find its harmonic conjugate.

Soln: u harmonic $\Rightarrow \nabla^2 u = 0$

$$\text{i.e. } u_{xx} + u_{yy} = 0.$$
$$\Rightarrow 6x + 2ax = 0 \Rightarrow \underline{a = -3}$$

To find the harmonic conjugate of u, we must find a v s.t. C-R. eqns hold.

$u_x = 3(x^2 - y^2) = v_y \Rightarrow$ upon integration w.r.t. y
$$v = 3x^2y - y^3 + f(x) \quad \text{—— (1)}$$

$u_y = -6xy = -v_x \Rightarrow$ upon integration w.r.t. x
$$v = 3x^2y + g(y) \quad \text{—— (2)}.$$

(1) & (2) $\Rightarrow v(x,y) = 3x^2y - y^3 + \text{Const}.$ #

Q.2) Comment if $f'(z)$ exists & if it does, find it.
$f(r,\theta) = \log r + i\theta$.

Soln:- CR conditions in polar form
$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

Here $u = \log r;\ v = \theta \Rightarrow u_r = \frac{1}{r} = \frac{v_\theta}{r}$ ✓
$$v_r = 0 = -\frac{u_\theta}{r} \ ✓$$

$\Rightarrow f$ is analytic everywhere
and $f'(z) \overset{HW}{=} e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) \equiv \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.$
$$= e^{-i\theta}/r = \frac{1}{z}. \quad \#.$$

Def^n (Singular point)

A singular point z_0 is a point where $f(z)$ fails to be analytic.

eg. (I) $f(z) = \frac{1}{z^2}$ has a singular point at $z=0$.

(II) $f(z) = \bar{z}$ is nowhere analytic & $\therefore$ has singular points everywhere on $\mathbb{C}$.

(III) $\dfrac{z}{z^4+1}$ is the ratio of 2 polynomials that are each entire f^ns.

So $\frac{z}{z^4+1}$ is analytic except when $z^4+1=0 \Rightarrow z^4 = -1$

$\Rightarrow z = e^{i(\pi/4 + n\pi/2)}$

$n = 0, 1, 2, 3$

or $z = \pm\frac{\sqrt{2}}{2}(1 \pm i)$

(IV) $e^{\frac{1}{z-1}}$ is analytic everywhere except at $z=1$.

(6.2) Multivalued f^ns, branch points & branch cuts.

Multi-valued f^ns are exactly what the name suggests; they take on multiple values at the same pt.

They are naturally introduced as inverses of single valued f^ns.

eg (I) $z = \omega^2$ is single valued.

But the inverse f^n $\omega = \sqrt{z} = z^{1/2}$ is multi-valued. Let us see why?

Let $z = re^{i\theta}$

Where $\theta = \theta_p + 2\pi n$; $\theta_p \in [0, 2\pi)$, $n \in I$.

$\omega = z^{1/2} = r^{1/2} e^{i\theta_p/2} e^{in\pi}$; $r^{1/2} = \sqrt{r} > 0$.

$$\therefore e^{in\pi} = \cos n\pi + i \sin n\pi = \begin{cases} -1 & ; n \text{ is odd} \\ 1 & ; n \text{ is even} \end{cases}$$

$$\therefore \omega = \begin{cases} -r^{1/2} e^{i\theta_p/2} & ; n \text{ is odd.} \\ r^{1/2} e^{i\theta_p/2} & ; n \text{ is even} \end{cases}$$

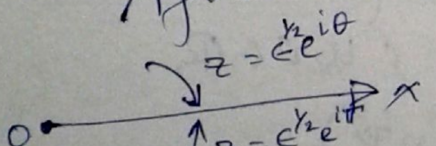
$\Rightarrow \omega$ is multivalued.

Now, let us follow a small circuit of radius $r = \epsilon$ around $z = 0$ — This is our choice

Also, $n = 0$ corresponds to $\theta_p = 0$ (principal branch) ← This is the origin of our circular path

& at this pt. we have $\omega = \sqrt{\epsilon}$

After one loop abt. the origin $\theta = \cancel{\theta_p}^{\,0} + 2\pi$ $(n=1)$



$$\omega = \sqrt{\epsilon}\, e^{i\pi} = -\sqrt{\epsilon}$$

$\therefore z = 0$ is called a branch pt.

Def^n (Branch pt.) A pt on $\mathbb{C}$ is a branch pt. if the multivalued f^n $\omega(z)$ is discontinuous upon traversing a small circuit around this pt.

Likewise by using the transformation $z = \frac{1}{t}$;
a branch pt. can be established around
$t = 0$ (or $z = \infty$).

Infact, $z = 0$ & $z = \infty$ are the <u>only branch</u>
pts. of $\omega = z^{1/2}$.

So in order to bypass <u>this multi-valuedness</u>
of the f^n; it is <u>often</u> helpful to introduce
a <u>branch cut</u> in order to artificially
create a region where ~~the~~ multivalued f^n
is <u>single-valued & continuous</u>.

for $\omega = z^{1/2}$; <u>Branch cut</u> is $\mathrm{Re}\, z \geq 0$ &

the <u>cut plane</u> is the branch cut
along w/ the branch pts. $z = 0$ &
$z = \infty$ removed i.e $\boxed{\mathbb{C} - \{z = 0, \infty, \mathrm{Re}\, z > 0\}}$

most precise def^n of cut plane!

often written as $\mathrm{Re}\, z > 0$ (simply). #

<u>eg (2)</u> <u>Branches of complex logarithm f^n.</u>

Let $z = e^{\omega}$ and $\omega = u + iv$; $u, v \in \mathbb{R}$
$= e^u e^{iv} = e^u(\cos v + i \sin v)$

Note if $v = 0$ we have $u = \log z$ whence $z \in \mathbb{R}$
i.e. $u = \log r$

But we have departed from the real line
many weeks back, so lets return to $\mathbb{C}$

Starting pt. $z = re^{i\theta_p}$; $\theta_p \in [0, 2\pi)$

* This choice is <u>not</u> unique & infact will determine location of branch pts.

Let $v = \theta = \theta_p + 2\pi n$

Whence $z = e^{u} e^{i(\theta_p + 2\pi n)}$

$$\therefore \omega = \log z = \log(e^{u} e^{iv})$$

$$= \log e^{u} + \log e^{i(\theta_p + 2\pi n)}$$

$$= u + i\theta_p + i2\pi n$$

$$\log z = \log r + i\theta_p + i2\pi n.$$

We saw earlier that is the real part of $\log z$.

$\therefore$ Complex logarithm differs from its real part by $i(\theta_p + 2\pi n)$

<u>Principal branch</u> corresponds to $n = 0$.

1) We may choose our starting pt. at $z = i$ corresponding to $\theta_p = \pi/2$ whence

$$\log z = \log 1 + i\pi/2 + 2\pi n i \quad ; \quad n = 0, \pm 1, \pm 2, \ldots$$

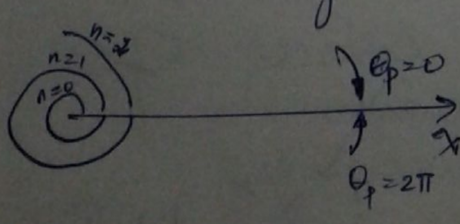
i.e. $\log z$ is <u>infinitely-valued</u> abt. $z = 0$ & $z = \infty$ (Take $z = 1/t$)

$\therefore$ B.Ps are $z = 0, \infty$

B.C. is $\mathrm{Re}\, z > 0$.

ii) Instead if we chose $z = r$ (real) as our starting pt. then $\theta_p = 0$

& $\log z = \log r + i2\pi n$



$n = 0$ is the principal branch; $\log z = \log r \Rightarrow z = r$.

$n = 1$, $\log z = \log r + 2\pi i$

$e^{()} \Rightarrow z = re^{i2\pi} \Rightarrow n = 1$ corresponds to 2π rotatn abt $O(0,0)$.

It is known that $\log z$ is analytic everywhere in the cutplane

$$\frac{d}{dz}\log z = \frac{1}{z}.$$

HW

eg (3) Try to find the cut plane for z^a by writing $z^a = e^{a\log z}$

eg (4) Find the cut plane for $\omega = \cos^{-1} z$.

HW

Complete this problem!

How will you approach this?

$$\because \cos\omega = z = \frac{e^{i\omega}+e^{-i\omega}}{2}$$

$$\Rightarrow \underbrace{e^{2i\omega} - 2z e^{i\omega} + 1 = 0}_{\text{roots of Quadratic eqn}}.$$

HINT

$$e^{i\omega} = z + (z^2-1)^{1/2}$$
$$= z + i(1-z^2)^{1/2}$$

$$\omega(z) = -i\log\{z + i(1-z^2)^{1/2}\}$$

By inspection we can see that $\omega(z)$ has "2" sources of multi-valuedness

$\log(\ )$ $\quad (\ )^{1/2}$

$\therefore$ We must expect $\omega(z)$ to be doubly infinitely valued!