Lecture: (15) DATE 20/3/2025

Eigenvalues & Eigenvectors

$$T(\vec{x}) = A\vec{x} = \lambda \vec{x} \quad \text{——— (1)}$$

ev – EV eq^n.

λ is the ev (eigenvalue) of A

$\vec{x}$ is the EV (eigenvector) of A.

Note :– if $\vec{x} = \vec{0}$ the eq^n (1) will be trivially satisfied; therefore, we exclude $\vec{0}$ as an EV. the eigenspace of A is $\{null(A-\lambda I)\} \cup \{\vec{0}\}$

Geometric meaning of $A\vec{x} = \lambda \vec{x}$

(1) if $\lambda \in \mathbb{R}$; then the transformation $T(\vec{x}) = A\vec{x} = \lambda \vec{x}$ is either a

DATE __/__/__

stretching of the vector $\vec{x}$ or a compression of the vector $\vec{x}$

② If $\lambda < 0$; then the transformation $A\vec{x} = \lambda\vec{x}$ is a "reversal" of $\vec{x}$ (in direction)

③ If $\lambda \in \mathbb{C}$ (esp. if $\lambda = \pm i$)

$T(\vec{x}) = A\vec{x} = \lambda\vec{x}$ is a 90° rotation of the vector $\vec{x}$.

④ If $\lambda \equiv 1$; then the transformation $T(\vec{x}) = A\vec{x} = \vec{x}$ is an invariant transformation; whence

$A \equiv I$ (identity matrix)

Algebraic meaning of $A\vec{x} = \lambda\vec{x}$

or equivalently $A\vec{x} = \lambda\vec{x}$

$$\Rightarrow A\vec{x} - \lambda\vec{x} = \vec{0}$$

$$\Rightarrow (A - \lambda I)\vec{x} = \vec{0}$$

$$\therefore \vec{x} \in \text{null}(A - \lambda I) \quad \text{—— } (2)$$

Further, since $\vec{x} \neq \vec{0}$

$(A - \lambda I)$ is not invertible

(b/c if it were then $\vec{x} = (A - \lambda I)^{-1}\vec{0} = \vec{0}$)

$$\therefore \det(A - \lambda I) = 0 \quad \text{—— } (3)$$

this is known as the Characteristic eqn. / Caley-Hamilton th^m.

* Eq(3) is often use to compute the evs of a matrix A by hand!

eg ①

Let us see how we can use eq(1) & eq(3) to compute the evs and EVs of a 3 × 3 matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 3 & 0 \\ 3 & 2 & 2 \end{pmatrix}$$

Let us use the ch. eqn (3) to find the λs (evs) of A

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ -3 & 3-\lambda & 0 \\ 3 & 2 & 2-\lambda \end{vmatrix}$$

$$= (1-\lambda)\{(3-\lambda)(2-\lambda) - 0\}$$

$$= (1-\lambda)(3-\lambda)(2-\lambda) = 0$$

* ⇒ $\lambda = 1, 2, 3$ are the evs of A

Infact for any diagonal/lower-Δ matrix/upper-Δ matrix, evs always appear along the diagonals

DATE

Now let us find the EVs corresponding to each ev $\{\lambda_1, \lambda_2, \lambda_3\}$

$\underline{\lambda_1 = 1}$

$$A\vec{x} = \lambda \bar{x}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ -3 & 3 & 0 \\ 3 & 2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x_1 \\ -3x_1 + 3x_2 \\ 3x_1 + 2x_2 + 2x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

or $$\begin{pmatrix} 3x_1 + 2x_2 + 2x_3 \\ -3x_1 + 3x_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} x_3 \\ x_2 \\ x_1 \end{pmatrix}$$

Chose $x_1 = \underline{1}$ arbitrarily.

$x_2 = \frac{3}{2}$ $(2x_2 = 3 \Rightarrow x_2 = 3/2)$

$$\vec{x_1} = \begin{pmatrix} 1 \\ 3/2 \\ -6 \end{pmatrix}$$

$2x_3 - x_3 = -3 - 2 \times \frac{3}{2}$

$\Rightarrow x_3 = -6$

$\lambda_2 = 2$

DATE ___/___/___

$A\vec{x} = \lambda_2 \vec{x}$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ -3 & 3 & 0 \\ 3 & 2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{pmatrix}$$

$\Rightarrow \quad x_1 = 2x_1 \Rightarrow x_1 = 0$

$-3x_1 + 3x_2 = 2x_2 \Rightarrow x_2 = 0$

$3x_1 + 2x_2 + 2x_3 = 2x_3$

$x_3 = 1$ (arbitrary)

$$\vec{x}_{\lambda_2} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$\lambda_3 = 3$

$$\begin{pmatrix} 1 & 0 & 0 \\ -3 & 3 & 0 \\ 3 & 2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3x_1 \\ 3x_2 \\ 3x_3 \end{pmatrix}$$

$$\vec{x}_{\lambda_3} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

$\therefore \quad x_1 = 0$

$x_2 = 1$ (arbitrary)

$2 + 2x_3 = 3x_3$

$\Rightarrow x_3 = 2$

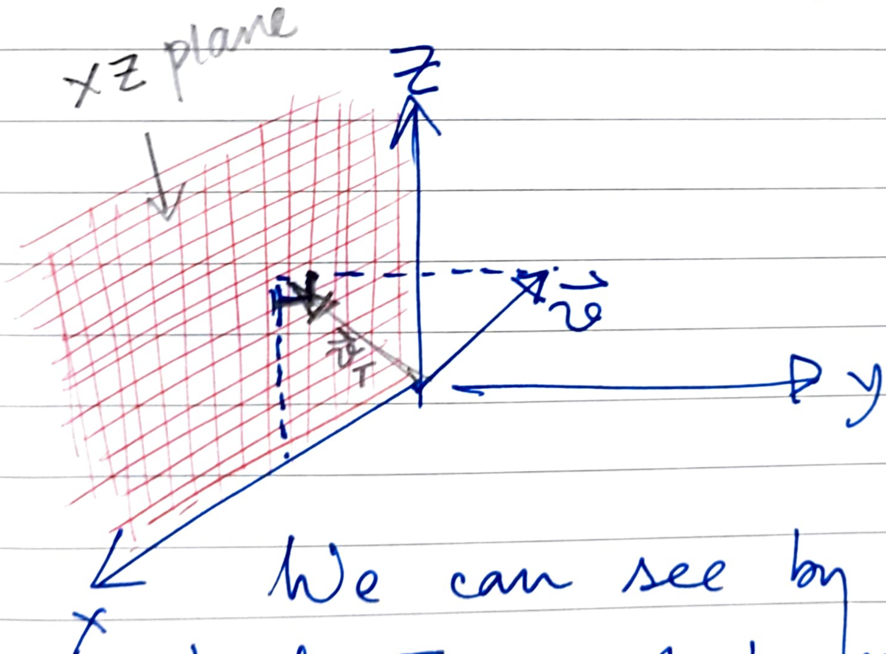
DATE

eg (2) What about the evs & EVs of an OG transformation?

Consider a vector $\vec{v} \in \mathbb{R}^3$

We want to project it OG^ly on the xz plane.

i.e. $\overset{\vec{v}}{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}} \xrightarrow{T} \overset{\vec{v}_T}{\begin{pmatrix} x_1 \\ 0 \\ x_3 \end{pmatrix}}$



We can see by inspection that T must have a matrix rep^n $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Clearly $A\vec{v} = \vec{v}_T$ (Note the r.h.s is $\vec{v}_T$ and NOT $\vec{v}$)

DATE / /

the evs are $\{1, 0\}$.

↑ repeated.

$\underline{\lambda = 1}$

$$A\bar{x} = \bar{x}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$x_1 = x_1$$
$$0 = x_2$$
$$x_3 = x_3$$

$\lambda = 1$ corresponds to the "retained" dim & hence the component corresponding to the collapsed dim (y) is 0

$$\therefore \vec{X}_{(\lambda=1)} = \begin{pmatrix} \alpha \\ 0 \\ \beta \end{pmatrix}$$ for α, β chosen arbitrarily

$\underline{\lambda = 0}$.

$$x_1 = 0$$
$$0 = 0$$
$$x_3 = 0$$

$\lambda = 0$ corresponds to the "Collapsed" dimension (here y-dim)

$$\vec{X}_{(\lambda=0)} = \begin{pmatrix} 0 \\ \delta \\ 0 \end{pmatrix}$$ where δ is arbitrary (say $\delta = 1$)

eg ③ "Projection Operators"

need not be always OG.

say we want (oblique projⁿ)

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \xrightarrow{T} \begin{pmatrix} 0 \\ x_1 + x_2 \end{pmatrix}$$

Choose

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$$

$$A\bar{x} = \lambda \bar{x}$$

$$\underbrace{\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ x_1 + x_2 \end{pmatrix}$$

the evs are $\{0, 1\}$.

<u>EVs</u> :- <u>$\lambda = 0$</u>. $\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\vec{X}_{(\lambda=0)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\quad v_1 + v_2 = 0 \Rightarrow v_1 = -v_2$

<u>$\lambda = 1$</u> $\quad 0 = v_1$ i.e. $\vec{X}_{(\lambda=1)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$v_1 + v_2 = v_2$

DATE / /

* Nevertheless, projⁿ operators (matrices $\mathbb{P}$) always have the following properties.

(I) $\mathbb{P}^2 = \mathbb{P}$ (idempotency)

(II) EVs of $\mathbb{P}$ are always 0 and 1.

Q) Why?

Ans)

$$\mathbb{P}^2 \vec{v} = \mathbb{P}\vec{v} = \lambda \vec{v}$$

$$\Rightarrow \mathbb{P}(\mathbb{P}\vec{v}) = \lambda \vec{v}$$

$$\Rightarrow \mathbb{P}(\lambda \vec{v}) = \lambda \vec{v}$$

$$\Rightarrow \lambda \mathbb{P}\vec{v} = \lambda \vec{v}$$

$$\Rightarrow \lambda(\lambda \vec{v}) = \lambda \vec{v}$$

$$\Rightarrow \lambda^2 = \lambda$$

$$\Rightarrow \lambda^2 - \lambda = 0.$$

$$\Rightarrow \lambda(\lambda - 1) = 0$$

$$\Rightarrow \lambda = 0, 1$$

eg (4) Q) What about eVs - EVs of reflection operations/transformations

Abt. x-axis

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \xrightarrow{T} \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix}.$$

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$\det(A - \lambda I) = 0.$

$\Rightarrow \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0.$

$$A\bar{x} = \lambda \bar{x}$$

$\lambda^2 = 1$

$\lambda = \pm 1$

eVs

$\lambda = 1$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 1\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{pmatrix} x_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{i.e. } \vec{X}_{(\lambda=1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\lambda = -1$

$$x_2 = -x_1$$

$$x_1 = -x_2$$

i.e. $\vec{X}_{(\lambda=-1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

EVs

eg (5)

DATE ___/___/___

A few more interesting matrix transformations ($\mathbb{P}$)

(i) Markov matrix : Each col^m of $\mathbb{P}$ adds to 1.

$\Rightarrow \lambda = 1$ is an ev.

(ii) P is singular, so $\lambda = 0$ is an ev

(iii) P is symmetric, so EVs
($p_{ij} = p_{ji}$)
$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \perp \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

if $A \in M_{n \times n}$

* (I) Product of evs (n of them) = det of A.

(II) $\lambda_1 + \lambda_2 + \dots + \lambda_n = \text{trace}(A)$
$= a_{11} + a_{22} + \dots + a_{nn}$