Time Series Models: Auto-Regressive model

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Moving Average model of order ∞

$MA(\infty)$:
Let ε_t be white noise. The $MA(\infty)$ model is defined as follows:
$Y_t = \mu + \sum_{j\geq 0} \psi_j \varepsilon_{t-j}; \quad \mu, \psi_j$ are constants.

$MA(\infty)$ is <u>stationary</u> if $\sum_{j\geq 0} |\psi_j| < \infty$ because all statistical moments $\mu, \gamma_0, \gamma_1, ...$ are finite and constant if the coefficients ψ_j are absolutely summable.

Further, $MA(\infty)$ is <u>ergodic</u> if $\sum_{j\geq 0} |\psi_j| < \infty$!

AR(1) model

Definition: $Y_t = c + \phi Y_{t-1} + \varepsilon_t$ where ϕ is a real constant.
Note: When $|\phi| \geq 1$, no stationary AR model exists! (ignore this case!)

When $|\phi| < 1$, we will show $AR(1)$ is complementary to $MA(\infty)$!
The stable solution to AR(1) defined above is obtained from

$$\begin{aligned} Y_t &= w_t + \phi w_{t-1} + \phi^2 w_{t-2} + ...; \quad w_t = c + \varepsilon_t \\ &= \frac{c}{1-\phi} + \varepsilon_t + \phi \varepsilon_{t-1} + \phi^2 \varepsilon_{t-2} + ... \end{aligned}$$

which is $MA(\infty)$ with $\mu = \frac{c}{1-\phi}$ and $\psi_j = \phi^j$.
Also, $\sum_{j\geq 0} |\psi_j| = \sum_{j \geq 0} |\phi|^j = \frac{1}{1-\phi} \infty$ when $|\phi < 1|$.

Auto-covariances of AR(1)

Exercise: Use the $MA(\infty)$ model to compute the following.

- $\gamma_0 = E(Y_t - \mu)^2 = \frac{\sigma^2}{1-\phi^2}$
- $\gamma_j = E(Y_t - \mu)(Y_{t-j} - \mu) = \frac{\phi^j}{1-\phi^2}\sigma^2$
- $\rho_j = \frac{\gamma_j}{\gamma_0} = \phi^j$

AR(p) model

$Y_t = c + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + ... + \phi_p Y_{t-p} + \varepsilon_t$ where
$\mu = \frac{c}{1-\phi_1-\phi_2-...-\phi_p}$, and
$\gamma_0 = \phi_1\gamma_1 + \phi_2\gamma_2 + .. + \phi_p\gamma_p$ and
$\gamma_j = \sum_{m\geq 1}^{p} \phi_m\gamma_{j-m} \quad \forall j > 1$ along with $\gamma_j = \gamma_{-j}$.

Yule Walker model

$$\rho_j = \sum_{m\geq 1}^{p} \phi_m\rho_{j-m}; \quad j = 1, 2, ...$$are the Yule Walker equations.

We will do a laboratory experiment using the Yule Walker model to forecast employment growth statistics!
https://235d9ee8-8e8c-4d7b-a842-264ad94cf102.filesusr.com/ugd/334434_25cd85ba56c24e8fbd245a9443e21d27.pdf

MA is complementary to AR

Likewise, $MA(1)$ is an $AR(\infty)$ process!

ARMA(p,q)

$Y_t = c+\phi_1 Y_{t-1}+\phi_2 Y_{t-2}+\ldots+\phi_p Y_{t-p}+\varepsilon_t+\theta_1\varepsilon_{t-1}+\theta_2\varepsilon_{t-2}+\ldots+\theta_q\varepsilon_{t-q}$.

Practise problem

Ques: Consider the MA(2) process $Y_t = \mu + \varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2}$ with $\theta_1 = 2/5$ and $theta_2 = -1/5$. Calculate all the autocorrelation functions.
Soln.: $\rho_0 = 1, \rho_1 = 4/15, \rho_2 = -1/6$ and $\rho_j = 0 \quad \forall j > 2$.