Complex Analysis

Motivating Questions

1. What is ~~the~~ meaning of i?

2. Is there a difference bet'n $\mathbb{R}^2$ and $\mathbb{C}$?

   (esp. b/c elements in each can be represented as an ordered pair).

RELATED

3. Are there complex numbers in higher dimensional space?

4. Can the field, $\mathbb{C}$ be (reduced) represented by (certain type of) Matrices?

   — Mathematics allows for multiple representations of the same entity.

Lecture (1) :- Complex Analysis 10/1/19

PMC - 103

Let two orthogonal (OG) vectors $\vec{e}_1$ and $\vec{e}_2$ be the bases of $\mathbb{R}^2$.

Any vector $\vec{r} = x\vec{e}_1 + y\vec{e}_2$; $|\vec{r}| = \sqrt{x^2+y^2}$

If $\vec{r}$ is multiplied by itself, a natural choice is

$$\vec{r}\vec{r} = \vec{r}^2 = |\vec{r}|^2$$

$$\text{i.e.} \ (x\vec{e}_1 + y\vec{e}_2)^2 = x^2+y^2$$

$$\Rightarrow x^2\vec{e}_1^{\,2} + y^2\vec{e}_2^{\,2} + xy(\vec{e}_1\vec{e}_2 + \vec{e}_2\vec{e}_1) = x^2+y^2$$

Above is satisfied if

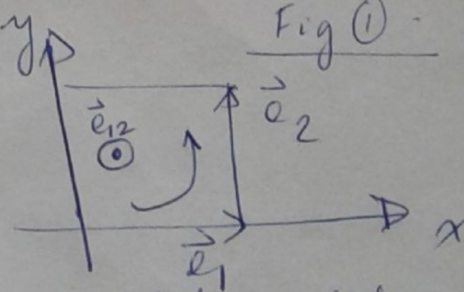
$$\left.\begin{array}{l} \vec{e}_1^{\,2} = \vec{e}_2^{\,2} = 1 \\ \& \ \underbrace{\vec{e}_1\vec{e}_2}_{} = -\vec{e}_2\vec{e}_1 \end{array}\right\} \begin{array}{l} \Rightarrow |\vec{e}_1| = |\vec{e}_2| = 1 \\ \Rightarrow \vec{e}_1 \perp \vec{e}_2 \end{array}$$

bi-vector (product of 2 vectors).

geometrical meaning } $\Rightarrow$ oriented plane area of the square w/ sides $\vec{e}_1$ and $\vec{e}_2$

$$\vec{e}_1\vec{e}_2 \equiv \vec{e}_{12}$$

Fig ①.



Clifford product of 2 vectors :-

$$\vec{a} = a_1\vec{e}_1 + a_2\vec{e}_2 \ ; \ \vec{b} = b_1\vec{e}_1 + b_2\vec{e}_2$$

$$\vec{a}\vec{b} = a_1b_1 + a_2b_2 + (a_1b_2 - a_2b_1)\vec{e}_{12} = \underbrace{\vec{a}\cdot\vec{b}}_{\text{scalar product}} + \underbrace{\vec{a}\wedge\vec{b}}_{\text{exterior product}}$$

thus one may form 4 bases

1 scalar

$\left.\begin{array}{l}\vec{e}_1 \\ \vec{e}_2\end{array}\right\}$ vectors.

$\vec{e}_{12}$ bivector.

& the associated algebra is called Clifford Algebra Cl_2 of $\mathbb{R}^2$.

b/c $\mathbb{R}^{(2)}$

In general $u = \underbrace{u_0 + u_1\vec{e}_1 + u_2\vec{e}_2 + u_{12}\vec{e}_{12}}_{\in Cl_2}$. linear combination

$\therefore Cl_2$ is a 4D real linear space w/ basis elements $\{1, \vec{e}_1, \vec{e}_2, \vec{e}_{12}\}$ which following the multiplication table

	$\vec{e}_1$	$\vec{e}_2$	$\vec{e}_{12}$
$\vec{e}_1$	1	$\vec{e}_{12}$	$\vec{e}_2$
$\vec{e}_2$	$-\vec{e}_{12}$	1	$-\vec{e}_1$
$\vec{e}_{12}$	$-\vec{e}_2$	$\vec{e}_1$	-1

Complex Numbers :-

$z = x + iy$

$\bar{z} = x - iy$ (reflection abt x-axis)

$$z^{-1} = \frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}$$

In polar form (Polar representation of complex no.s)

$x = r\cos\varphi$
$y = r\sin\varphi$

$z = x + iy = r(\cos\varphi + i\sin\varphi)$; $\varphi \in \mathbb{R}$ is called phase $\angle^{le}$ or arg. of z

$|z| = \sqrt{x^2+y^2} = \sqrt{z\bar{z}}$

Let $z_1 = r_1(\cos\varphi_1 + i\sin\varphi_1)$
$z_2 = r_2(\cos\varphi_2 + i\sin\varphi_2)$

$|z_1 z_2| = |z_1||z_2|$

We will revisit later that exponential f^n can be defined everywhere in the complex plane by

$$e^z = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^k}{k!} + \cdots = \exp(z)$$

If we expand $\cos\varphi$ & $\sin\varphi$ as a series then

$$e^{i\varphi} = \cos\varphi + i\sin\varphi \quad \text{(Euler's formula)}$$

$\Rightarrow \boxed{z = re^{i\varphi}}$ polar form of complex no.s.

** $z_1 z_2 = (r_1 r_2) e^{i(\varphi_1+\varphi_2)}$
$z^n = r^n e^{in\varphi}$

Matrix representation of Complex Numbers.

Complex no.s were constructed as ordered pairs of real numbers.

$$\therefore \quad z = x + iy \text{ in } \mathbb{C} \overset{?}{\equiv} \begin{pmatrix} x \\ y \end{pmatrix} \text{ in } \mathbb{R}^2$$

this makes explicit the real linear structure on $\mathbb{C}$.

In the same spirit, the product of 2 Complex no.s $c = a + ib$ and z.

$$cz = (ax - by) + i(bx + ay) \text{ can}$$

be thought to be equivalent to.

b/c

$$cz \equiv \begin{pmatrix} ax - by \\ bx + ay \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} a & -b \\ b & a \end{pmatrix}}_{\equiv c} \underbrace{\begin{pmatrix} x & -y \\ y & x \end{pmatrix}}_{\equiv z} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

This representation of $\mathbb{C}$ by Mat(2, $\mathbb{R}$) is not unique!!

Thus we may consider representing complex no.s by certain real 2×2 matrices in Mat(2, $\mathbb{R}$):

$$\mathbb{C} \longrightarrow \text{Mat}(2, \mathbb{R}) \,; \quad a + ib \longrightarrow \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

$\mathbb{C}$	Mat(2, $\mathbb{R}$)
1	$\to I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
i	$\to J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

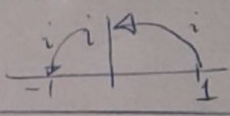
Geometrical interpretation of $i = \sqrt{-1}$

<u>Goal</u> :- In this section we shall study the introduction of complex no.(s) by means of Cl_2 (Clifford algebra of Euclidean plane $\mathbb{R}^2$)

this approach assigns <u>two</u> different meaning to $i = \sqrt{-1}$

→ (1) an oriented plane area in $\mathbb{R}^2$

(this we have already seen in last lecture) ✓ (ii) $\pi/2$ rotation in $\mathbb{R}^2$.



$i(i \cdot 1) = i^2 = -1$
$\Rightarrow i$ is a $\pi/2$ rot^n

Recall from pg(1) of Lecture notes (1) (i.e. pg(1) of this set of lecture notes)

$$\vec{e}_1^{\,2} = \vec{e}_2^{\,2} = 1 \quad \text{and} \quad \vec{e}_1\vec{e}_2 = -\vec{e}_2\vec{e}_1$$

$$\Rightarrow (\vec{e}_1\vec{e}_2)(\vec{e}_1\vec{e}_2) = \vec{e}_1(\vec{e}_2\vec{e}_1)\vec{e}_2 = \vec{e}_1(-\vec{e}_1\vec{e}_2)\vec{e}_2 = -\vec{e}_1^{\,2}\vec{e}_2^{\,2} = -1$$

$$(\vec{e}_1\vec{e}_2)(\vec{e}_1\vec{e}_2) = (\vec{e}_1\vec{e}_2)^2 = (\vec{e}_{12})^2$$

i.e. $(\vec{e}_{12})^2 = -1 \Rightarrow \boxed{\vec{e}_{12} = \sqrt{-1}}$

$\vec{e}_{12}$ is neither a scalar (b/c sq. of scalar > 0) nor a vector (obviously)

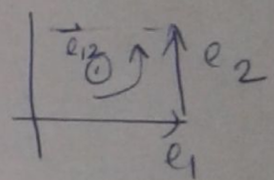
$\therefore \quad i = \sqrt{-1}$

We could write $\quad i \equiv \vec{e}_{12}$

$\therefore$ Comparing w/ fig(1) in pg(1) here,

** $\boxed{i \text{ means an oriented plane area in } \mathbb{R}^2}$

$\mathbb{C}$ vs $\mathbb{R}^2$



So if we write $\quad z = x + iy \in \mathbb{C}$

it means $\equiv$ ly

$$z = \underbrace{x}_{\text{scalar}} + \underbrace{\vec{e}_{12}\, y}_{\text{bivector}}$$

i.e. $\mathbb{C}$ is spanned by $\{1, \vec{e}_{12}\}$

& constitutes the complex plane.

in fact, $\mathbb{C} = Cl_2^+$

Whereas

$\mathbb{R}^2$ is spanned by $\{\vec{e}_1, \vec{e}_2\} = \{(1,0), (0,1)\}$

and is a vector plane

infact, $\mathbb{R}^2 = Cl_2^-$

Q) How does the Clifford algebra help us to interpret i as a $\pi/2$ rotation?

Ans) If we follow from above that

$$i = \vec{e}_{12} \text{ and}$$

Consider $\vec{r} = x\vec{e}_1 + y\vec{e}_2$

$$\vec{r}\vec{e}_{12} = (x\vec{e}_1 + y\vec{e}_2)\vec{e}_{12}$$

$$\underset{\substack{\text{table of}\\ \text{multiplication}\\ \text{from pg (2)}}}{=} x\vec{e}_1\vec{e}_{12} + y\vec{e}_2\vec{e}_{12}$$

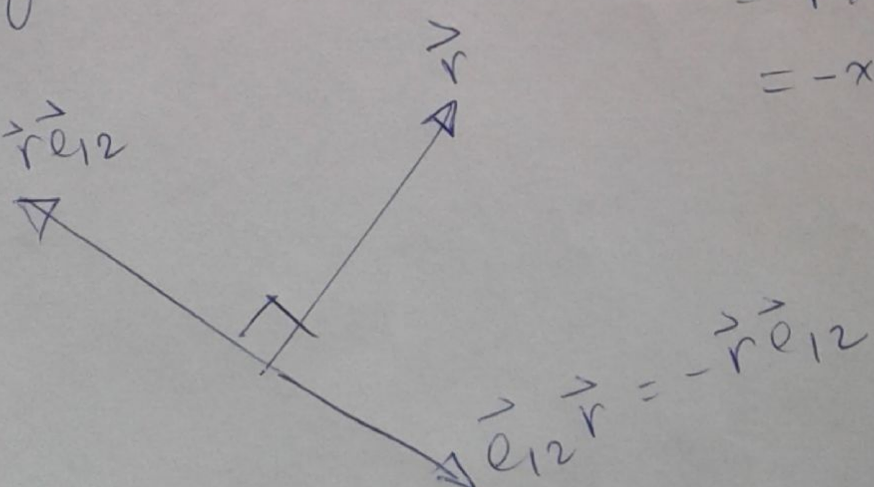
$$= x\vec{e}_2 - y\vec{e}_1 = \vec{r}\,'$$

Likewise $\vec{e}_{12}\vec{r} = y\vec{e}_1 - x\vec{e}_2$

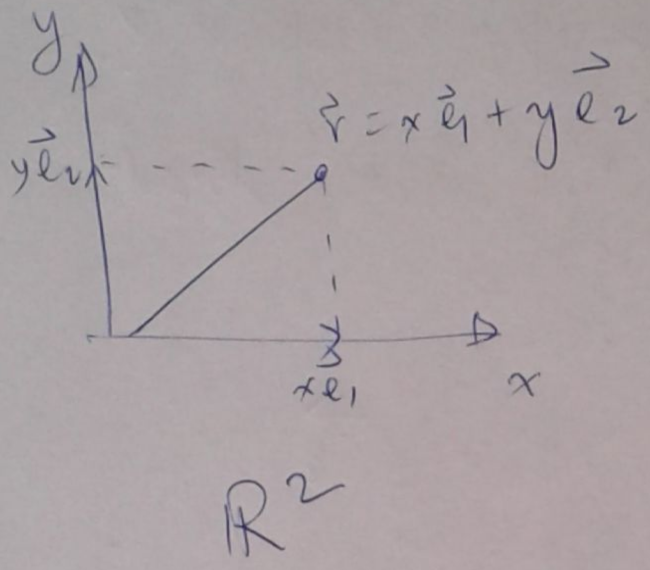
Clearly $\vec{r}\,' \perp \vec{r}$ b/c $\langle \vec{r}, \vec{r}\,' \rangle$

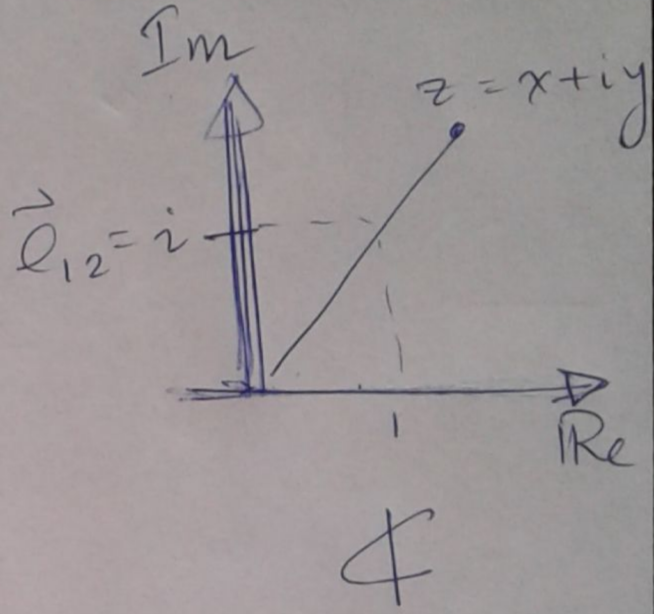
$= \vec{r} \cdot \vec{r}\,'$

$= -xy + xy = 0$



i.e. $\vec{e}_{12}$, and equivalently, i is a $\pi/2$ Rotor!


y
$\vec{r} = x\vec{e}_1 + y\vec{e}_2$
$y\vec{e}_2$
$x e_1$
x
$\mathbb{R}^2$


Im
$z = x + iy$
$\vec{e}_{12} = i$
Re
$\mathbb{C}$

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Lecture (2) :- Geometry of Complex numbers.

Recall from previous lecture :-

(I) i means $\pi/2$ rotation

(II) i also means a bi-vector (oriented plane area w/ sides $\vec{e}_1$ and $\vec{e}_2$)

(III) $\mathbb{C} \not\equiv \mathbb{R}^2$

$\mathbb{C}$: sum of scalar & bivector

$\mathbb{R}^2$: sum of 2 vectors.

Here sum means linear combination.

(IV) Representations of $z \in \mathbb{C}$

1) $x + iy$ (Cartesian form)

11) $re^{i\theta}$ (polar ")

111) $\begin{pmatrix} x & -y \\ y & x \end{pmatrix}$ (Matrix form)

In this lecture, we will discuss further about the geometry of the complex plane.

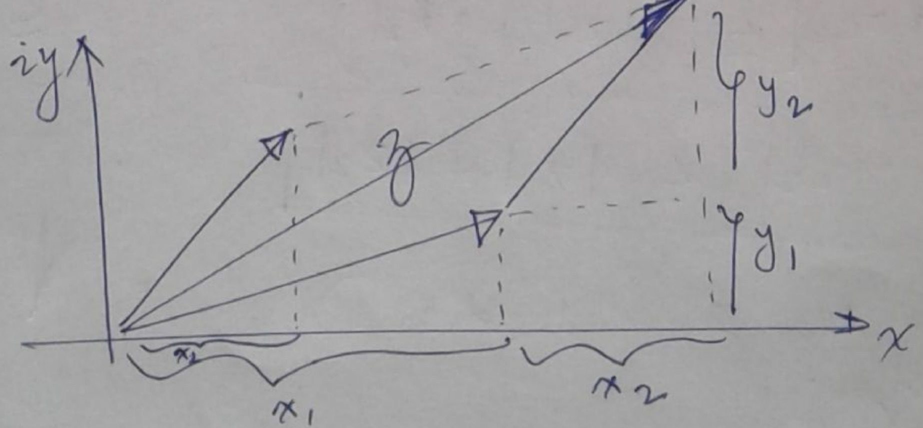
(2.1) Parallelogram Law, Triangle inequality.

Geometrically speaking, addition of 2 complex no.s is equivalent to that of parallelogram law of vectors.

Why? $z_1 = x_1 + iy_1$
$z_2 = x_2 + iy_2$

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2) = z$$

$$|z_1 + z_2| = \sqrt{(x_1+x_2)^2 + (y_1+y_2)^2} = |z|$$



Triangle Inequality.

$$\big||z_1| - |z_2|\big| \le |z_1 + z_2| \le |z_1| + |z_2|$$

Proof :- Recall $|z|^2 = z\bar{z}$ (If you are unsure, convince ~~this~~ yourself why this is true (HW))

$$\therefore |z_1+z_2|^2 = (z_1+z_2)(\overline{z_1+z_2}) = (z_1+z_2)(\bar{z}_1+\bar{z}_2)$$
$$= z_1\bar{z}_1 + z_2\bar{z}_2 + z_1\bar{z}_2 + \bar{z}_1 z_2$$
$$= |z_1|^2 + |z_2|^2 + 2\,\mathrm{Re}(z_1\bar{z}_2)$$

Why?

Again $z_1\bar{z}_2 + \bar{z}_1 z_2 = (x_1+iy_1)(x_2-iy_2) + (x_1-iy_1)(x_2+iy_2)$
$$= (x_1x_2 + x_1x_2) + (y_1y_2 + y_1y_2)$$
$$= 2\,\mathrm{Re}(z_1\bar{z}_2).$$

$z_1\bar{z}_2 = (x_1x_2 + y_1y_2) + i(x_2y_1 - x_1y_2)$ —— (i)

$$\therefore |z_1+z_2|^2 - (|z_1|^2 + |z_2|^2) = 2\,\mathrm{Re}(z_1\bar{z}_2)$$

$$\Rightarrow |z_1+z_2|^2 - \left(|z_1|^2 + |z_2|^2 + 2|z_1||z_2| - 2|z_1||z_2|\right) = 2\,\mathrm{Re}(z_1\bar{z}_2)$$

$$\Rightarrow |z_1+z_2|^2 - (|z_1| + |z_2|)^2 = 2\{\mathrm{Re}(z_1\bar{z}_2) - |z_1||z_2|\} \le 0$$

b/c $|z_1||z_2| = \sqrt{x_1^2+y_1^2}\sqrt{x_2^2+y_2^2}$
$= \sqrt{(x_1x_2)^2 + (y_1y_2)^2 + (x_1y_2)^2 + (x_2y_1)^2}$

& then compare w/ eq. (i)

$$\Rightarrow |z_1+z_2|^2 \le (|z_1|+|z_2|)^2$$

$$\Rightarrow \boxed{|z_1+z_2| \le |z_1| + |z_2|}$$

this proves the right hand inequality.

In order to prove the left hand inequality, we must redefine terms.

$$\omega_1 = z_1 + z_2, \quad \omega_2 = -z_2 \quad \text{—— (11)}$$

Now using the result ~~$|z_1+z_2| \le |\omega_1+\omega_2|$~~

$$|z_1 + z_2| \le |z_1| + |z_2|$$

$$\Rightarrow |\omega_1| \le |\omega_1 + \omega_2| + |-\omega_2|$$

$$\Rightarrow |\omega_1| - |\omega_2| \le |\omega_1 + \omega_2|$$

$$\Rightarrow \big||\omega_1| - |\omega_2|\big| \le |\omega_1 + \omega_2| \quad \text{if } |\omega_1| \ge |\omega_2|$$

If $|\omega_1| < |\omega_2|$; simply swap the definitions in (11) & the result would follow.

#

Generalization of $\triangle$ - inequality.

$$\left|\sum_{j=1}^{n} z_j\right| \le \sum_{j=1}^{n} |z_j| .$$

(2.2) Elementary functions: definitions, topology, properties.

Def^{n}s:

(i) A neighborhood of a point z_0 is the set of points z s.t. $|z - z_0| < \epsilon$ for some small $\epsilon > 0$.

(ii) Annulus :- $r_1 < |z - z_0| < r_2$
$z_0 \to$ center

(iii) A point z_0 of a set of points S is called an interior point of S if $\exists$ a neighborhood of z_0 that is contained entirely w/in S.

(iv) The set S is open if all pts. of S are interior points.

(v) A point z_0 is a boundary point of S if every neighborhood of $z = z_0$ contains atleast one point in S & atleast 1 pt. not in S.

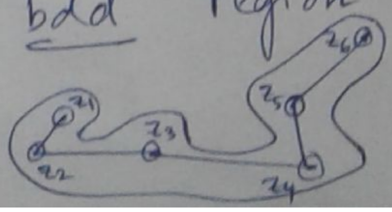
(vi) A set consisting of all points of an open set & none / some or all of its bdy pts is a region.

(vii) An open region is said to be bounded if $\exists$ a constant $M > 0$ s.t. all points z of the region satisfy $|z| \leq M$.

(viii) A region is closed if it contains all its boundary points.

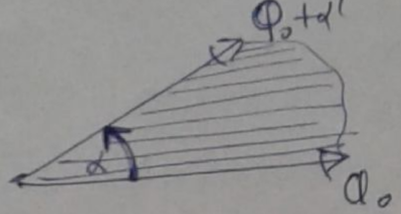
(ix) A closed & bdd region is compact.

(x) Connected region



(XI) A connected open region is called a domain.

eg. $S = \{ z = re^{i\varphi} : \varphi_0 < \arg z < \varphi_0 + \alpha \}$



If R is a region
$\overline{R}$ is its closure.
If R is closed then, $R = \overline{R}$.

* Up until now, our notion of function demanded single-valuedness. In this course we will discuss about multi-valued f^n.

eg of f^ns :-

(I) power f^n :- $f(z) = z^n$, $n = 0, 1, 2 \ldots$

(II) polynomial f^n :- $P_n(z) = \sum_{j=0}^{n} a_j z^j$; $a_j \in \mathbb{C}$

Domain of $P_n(z)$ is $\mathbb{C}$.

(III) Rational f^n $R(z) = \dfrac{P_n(z)}{Q_m(z)}$; $Q_m(z) = \sum_{j=0}^{m} b_j z^j$

Domain of $R(z)$ is $\mathbb{C} - \{ z : Q_m(z) = 0 \}$.
or $\mathbb{C} \setminus \{ z \mid Q_m(z) = 0 \}$.

(IV) Complex f^n

when $z = x+iy$, $f(z)$ is complex & written as $f(z) = u(x,y) + i v(x,y)$

eg. a) $w = z^2 = (x+iy)^2 = x^2 - y^2 + 2ixy$

which implies $u(x,y) = x^2 - y^2$

$v(x,y) = 2xy$.

b) exponential f^n

$$e^z = e^{x+iy} = e^x \underbrace{e^{iy}}_{\cos y + i\sin y}$$

w/ properties $e^{z_1+z_2} = e^{z_1}e^{z_2}$

$(e^z)^n = e^{nz}$; $n = 1, 2, \ldots$

$|e^z| = e^x$

$\overline{(e^z)} = e^{\bar{z}} = e^x(\cos y - i\sin y)$

c) <u>trigonometric f^ns.</u>

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \; ; \; \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\tan z = \frac{\sin z}{\cos z} \; , \; \sec z = \frac{1}{\cos z}$$

$$\cot z = \frac{\cos z}{\sin z} \; , \; \csc z = \frac{1}{\sin z}$$

All usual trigonometric properties hold

$\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$

$\sin^2 z + \cos^2 z = 1$.

d) <u>Hyperbolic f^ns</u> :- Likewise, $\sinh z = \frac{e^z - e^{-z}}{2}$ (odd part of e^z)

$\cosh z = \frac{e^z + e^{-z}}{2}$ (even part of e^z), $\operatorname{csch} z = \frac{1}{\sinh z}$

$\tanh z = \frac{\sinh z}{\cosh z}$; $\coth z = \frac{\cosh z}{\sinh z}$; $\operatorname{sech} z = \frac{1}{\cosh z}$, w/ properties $\cosh^2 z - \sinh^2 z = 1$

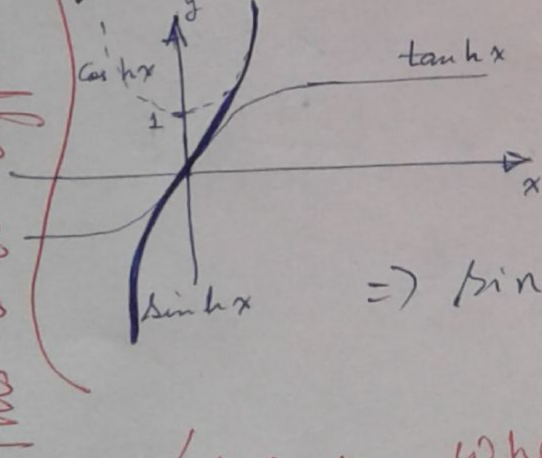
$$\sinh iz = i\sin z$$
$$\sin iz = i\sinh z$$
$$\cosh iz = \cos z$$
$$\cos iz = \cosh z.$$

This is a typical interview question!

NOTE :- So far we have said that trigonometric f^ns of complex no.s behave analogously like their real counterparts.

But there is a major fundamental diff.

$$\sin z = \sin(x+iy) = \sin x \cos iy + \cos x \sin iy$$
$$= \sin x \cosh y + i\cos x \sinh y$$



$\longrightarrow \infty$ as $y \to \infty$ b/c $\cosh y, \sinh y \to \infty$

$\Rightarrow \sin z$ is not bdd. but $|\sin x| \le 1$.

Later on when we discuss the Liouville's th^m, we will see why $\sin z$ could not have been bdd b/c for $f : \mathbb{C} \to \mathbb{C}$ differentiable & bdd $\Rightarrow f$ is const. & $\sin(z)$ is certainly not constant.

Power series representation of f^ns.

We will have a whole chapter devoted to power series of f^ns in $\mathbb{C}$; but as an introduction it is worthwhile to note some of the similarity w/ the case of reals

All elementary f^n's introduced earlier have power series representⁿ!

$$f(z) = \underbrace{\sum_{j=0}^{\infty} a_j (z - z_0)^j}$$; a_j & z_0 are constants.

for this to be true, convergence of the sum is crucial.

Ratio test. $\Rightarrow$ Conv. is guaranteed by

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| |z - z_0| < 1$$

i.e. the sum converges inside the circle $|z - z_0| = R$ where $R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$ is the radius of convergence.

Why is this the case?

the ratio test states, that for convergence,

$$\lim_{n\to\infty} \left|\frac{a_{n+1} (z - z_0)^{n+1}}{a_n (z - z_0)^n}\right| < 1$$

Equivalently, $|z - z_0| < \dfrac{1}{\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|} = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = R.$

If $R = \infty$, then the series conv. $\forall$ finite z.
& if $R = 0$, then the series conv. only for $z = z_0$.

Lecture (3): Mapping & Projections. 14/11/19

Stability:-

Often in dynamical systems, we find solutions that are of the form proportional to e^{zt}; $t>0$, $z \in \mathbb{C}$

Solutions are:-

Unstable $\rightarrow$ if $\mathrm{Re}(z) > 0$ b/c then $|e^{zt}| \to \infty$ as $t \to \infty$ (time)

marginally Stable $\rightarrow$ if $\exists$ no values of z for which $\mathrm{Re}(z) > 0$ but there exist some z s.t. $\mathrm{Re}(z) = 0$ (for which solutions are obviously bdd in t).

Stable (damped) $\rightarrow$ if for all values of z, $\mathrm{Re}(z) < 0$ (s.t. $|e^{zt}| \to 0$ as $t \to \infty$).

(2.3)

Mappings

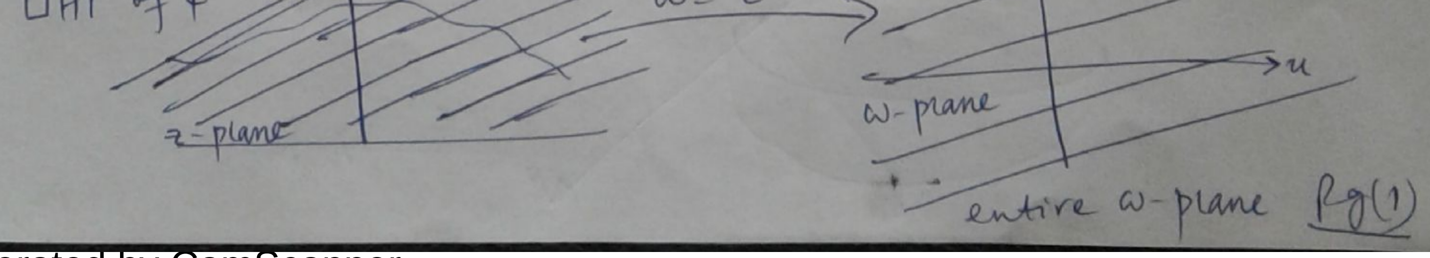
Just like in the case of real euclidean space, it may be convenient to do/perform certain mathematical analysis by transforming the variables from one domain to another.

eg 1 Consider the map $\omega = z^2$.

Let $z = re^{i\theta}$; $0 \le \theta < \pi$

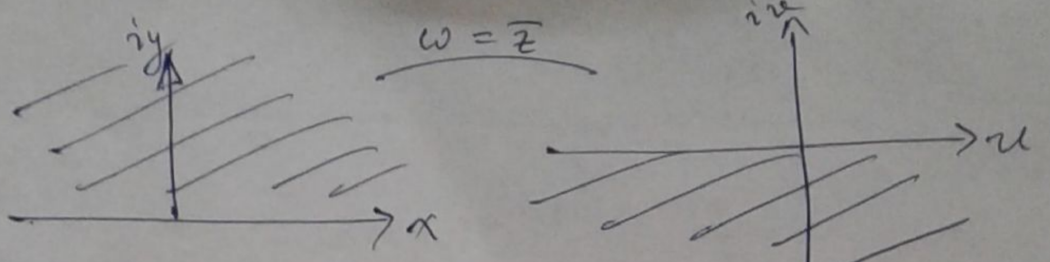
$\omega = r^2 e^{2i\theta} \equiv Re^{i\phi}$ where $R = r^2$, $\phi = 2\theta$

$\Rightarrow 0 \le \phi < 2\pi$



Pg(1)

eg (2)



$z = x + iy,\ y > 0 \longrightarrow w = \bar{z} = x - iy$

where $u = x$

$v = -y$

Point at infinity (∞ or z_∞)

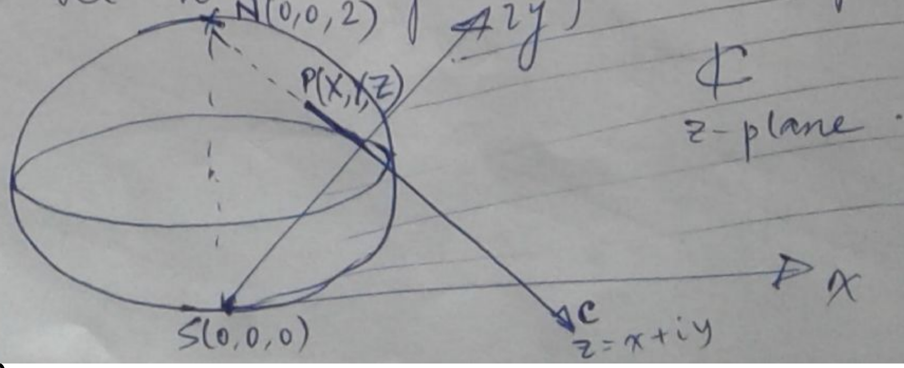
It is often useful to add the pt z_∞ to $\mathbb{C}$ & define the neighborhood of such a pt as all pts z s.t. $|z| > \frac{1}{\epsilon}$ $\forall\ \epsilon > 0$ (sufficiently small)

One convenient way of defining the pt. at ∞ is by considering the substitution $z = \frac{1}{t}$ & then say $t = 0 \Leftrightarrow z_\infty$. By doing this, we can use the def^n. of neighborhood provided earlier i.e. $|z - z_0| < \epsilon$.

The complex plane ($\mathbb{C} \cup z_\infty$) is called the extended complex plane. (Compactification of $\mathbb{C}$)

(2.4) Stereographic projections.

Consider a unit sphere sitting on top of the complex plane w/ the south pole of the sphere located at the origin of the z-plane



In this section, we plan to show how the extended complex plane can be mapped onto the surface of a sphere w/

South pole, $S(0,0,0) \equiv$ origin $(0,0)$ of complex plane

& North pole, $N(0,0,2) \equiv z_\infty$ on $\mathbb{C}$.

All other pts. have a desirable 1-1 correspondence by using the following construction:—

Connect $z = x+iy$ on $\mathbb{C}$ w/ North pole (NP) by a straight line as shown in previous figure. This line intersects the sphere at $P(X, Y, Z)$ s.t. $z = x+iy \overset{\text{uniquely}}{\Longleftrightarrow} P(X, Y, Z)$

This construction is called the <u>Steregraphic projection</u>. The compactification of $\mathbb{C}$ becomes visually (intuitively) clear by this construction.

<u>Details of the construction</u>

$N(0,0,2) = NP$

$P(X, Y, Z)$ on sphere surface.

$C(x, y, 0)$ on $\mathbb{C}$.

$\because$ they lie on a straight line,

$$\overrightarrow{PN} = \lambda \overrightarrow{CN} \; ; \; \lambda \in \mathbb{R}, \lambda \neq 0.$$

i.e. $(X, Y, Z-2) = \lambda(x, y, -2)$

$$\Rightarrow X = \lambda x, \; Y = \lambda y, \; Z = 2 - 2\lambda$$

Must satisfy eqn. of the sphere:—

$$x^2 + y^2 + (z-1)^2 = 1$$

$$\Rightarrow \lambda^2 x^2 + \lambda^2 y^2 + (2 - 2\lambda - 1)^2 = 1$$

$$\Rightarrow \lambda^2 (x^2 + y^2 + 4) - 4\lambda = 0$$

$$\Rightarrow \lambda \neq 0, \quad \lambda = \frac{4}{|z|^2 + 4}$$

$\therefore$ the unique correspondence of $z = x + iy$ on the surface of the sphere is given by

$$X = \frac{4x}{|z|^2+4}, \quad Y = \frac{4y}{|z|^2+4}, \quad Z = \frac{2|z|^2}{|z|^2+4}$$

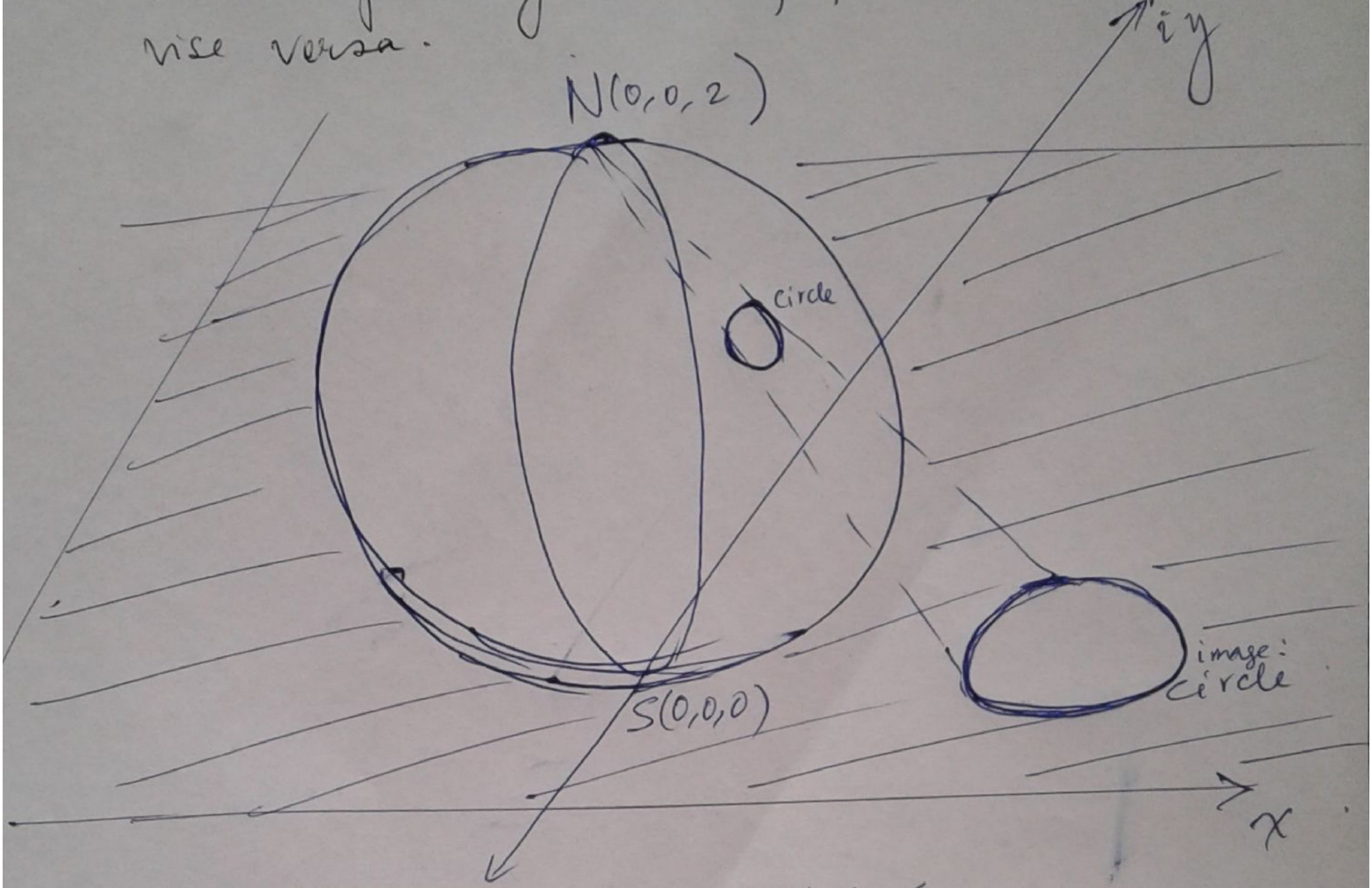
$$z = 0 \Rightarrow \begin{cases} Z = 0 \\ Y = 0 \\ X = 0 \end{cases} \text{ the SP.}$$

$$\text{and } |z| \to \infty \quad \left.\begin{matrix} X, Y \to 0 \\ Z \to 2 \end{matrix}\right\} \text{ NP.}$$

Likewise, given any $P(X, Y, Z)$;
the uniquely determined corresponding pt. on $\mathbb{C}$ is

$$x = \frac{2X}{2-Z}, \quad y = \frac{2Y}{2-Z}, \quad \lambda = \frac{2-Z}{2}$$

* The stereographic projection maps any locus of points in the complex plane onto a corresponding locus of pts. on the sphere & vise versa.



Note :- i) Circle passing through N & S is a st. line on $\mathbb{C}$.

ii) Circle on the surface of sphere is circle on $\mathbb{C}$.

We lose Euclidean geometry on the sphere but this may actually be desirable in many engineering & scientific problems.

— x —