Lecture (20) Conformal Mappings & Applications 24/4/19

§ (20.1) Introductory note.

Many mathematical, physical & engineering problems involve solving the Laplace eqn

$$\phi_{xx} + \phi_{yy} = 0 \quad \text{— (1)}$$ in a certain region

D of the complex plane. The above eqn. is supplemented by bdy conditions on ∂D.

Recall that the real & imaginary parts of an analytic f^n satisfies eqn. (1); it follows that solving the above problem reduces to finding a f^n that is analytic in D & satisfies the prescribed bdy condn. on ∂D.

The soln. of this problem turns out to be much simpler if D is the UHP or the unit disk.

* According to a celebrated theorem first discussed by Riemann; if D is a simply connected region D, that is not the entire complex z-plane; then $\exists$ an analytic f^n $f(z)$ such that $w = f(z)$ transforms D onto the UHP.

Unfortunately this thm. does not provide a constructive approach for finding $f(z)$.

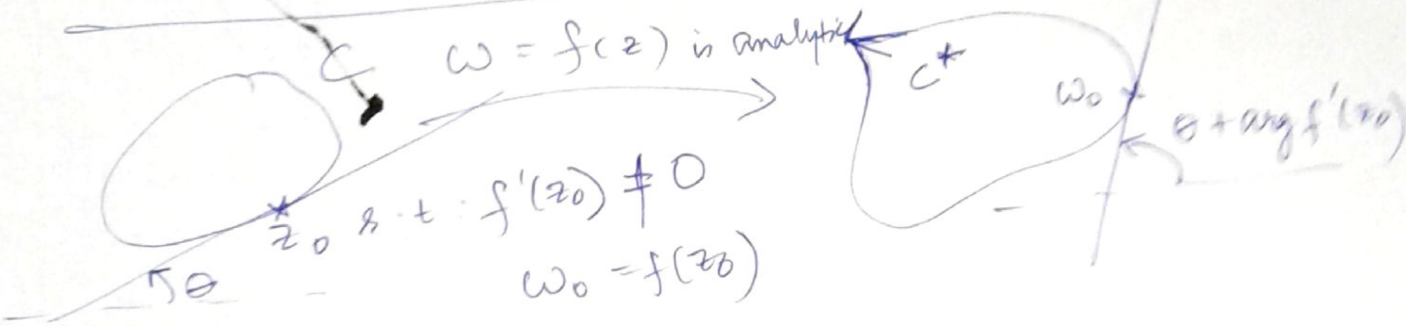
Spl. cases.

* From polygon domains $\xrightarrow{\text{Schwarz–Christoffel transformation}}$ UHP

* Bilinear transformations

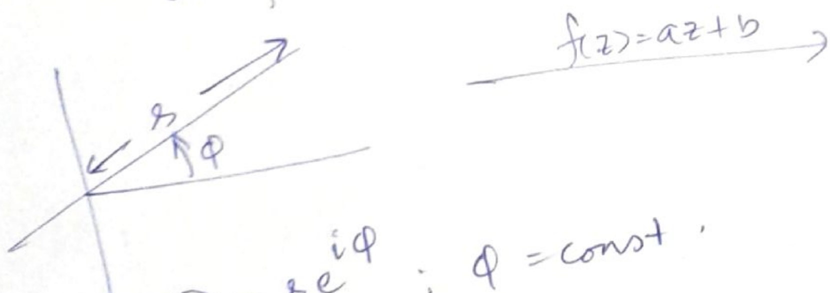
*  $\xrightarrow[\text{modular } f^n\text{s}]{\text{Schwarzian } f^n \text{ \& elliptic}}$ UHP.

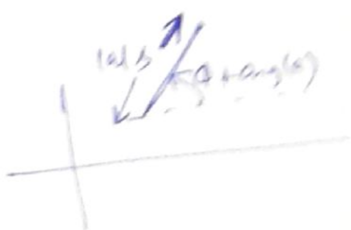
§ (20.2) Conformal Transformations.



eg C is a st. line

$\omega = f(z) = az + b$; $a, b \in \mathbb{C}$





$z(s) = s e^{i\varphi}$; $\varphi = \text{const}$.

$$\omega(s) = f(z(s)) = a\,z(s) + b$$

$$= |a| e^{i\arg(a)} s e^{i\varphi} + b$$

$$= |a| s\, e^{i(\varphi + \arg(a))} + b.$$

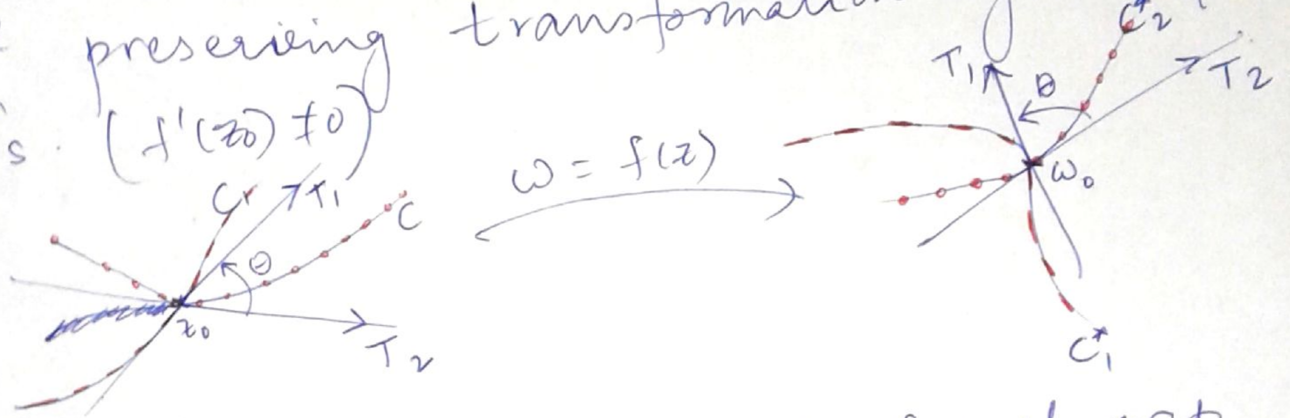
$\therefore f'(z(s)) = a \Rightarrow \arg(a) = \arg(f'(z))$

$$\therefore \omega(s) = |a| s\, e^{i(\varphi + \arg(f'(z)))} + b$$

HW :- Read pg (312-314) to convince yourself that under the analytic transformation $f(z)$; the directed tangent to any curve through z_0 is rotated by an angle $\arg(f'(z_0))$ in the counter-clockwise dir^n.

Def^n (Conformal transformation)

$\angle$le preserving transformation by analytic f^{n}s. $(f'(z_0) \neq 0)$



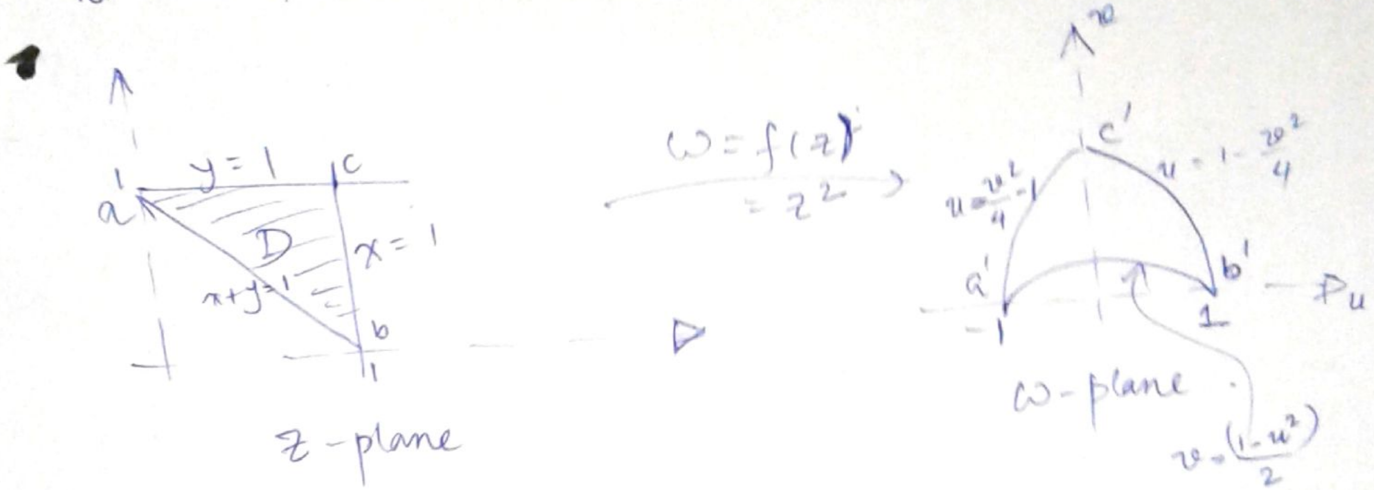
then $w = f(z)$ is a conformal map & this is an immediate consequence of the above result.

Th^m (20.1) Let $f(z)$ be analytic & not constant in a domain D of the complex z-plane. For any pt $z \in D$ for which $f'(z) \neq 0$; $w = f(z)$ is a conformal map i.e. it preserves angles bet'n 2 differentiable arcs.

A conformal mapping, in addition to preserving angles, has the property of magnifying pg(3) distances near z_0 by the factor $|f'(z_0)|$.

e.g. Let D be a triangular region bdd by $x=1$, $y=1$ and $x+y=1$.

the image of D under the transformation $\omega = z^2$ is given by the curvilinear Δ $a'b'c'$ as shown below



$z = x + iy$

$z^2 = (x+iy)(x+iy)$

$= \underbrace{x^2 - y^2}_{u} + i\,\underbrace{2xy}_{v}$

$x = 1 \xrightarrow{\omega = z^2} u = 1 - y^2$

$v = 2y$

$\Rightarrow u = 1 - \frac{v^2}{4}$

$y = 1 \xrightarrow{\omega = z^2} u = x^2 - 1$

$v = 2x$

$u = \frac{v^2}{4} - 1$

$x + y = 1 \xrightarrow{\omega = z^2} v = \frac{1 - u^2}{2}$

B/c $f'(z) = 2z$ & $z = 0$ is outside D in the z-plane $\Rightarrow$ the mapping is conformal.

§(20.3) Critical points & Inverse mappings.

Def^n Critical pt.

If $f'(z_0) = 0$; then the analytic transformation $f(z)$ ceases to be conformal. Such a pt. z_0 is called a critical pt. of f.

Intuitive Reasoning (What happens at a critical pt?).

Let $\delta z = z - z_0$; $z \approx z_0$

$$f'(z_0) = f''(z_0) = \cdots = f^{(n-1)}(z_0) = 0; \; f^{(n)}(z_0) \neq 0.$$

then Taylor expanding abt z_0.

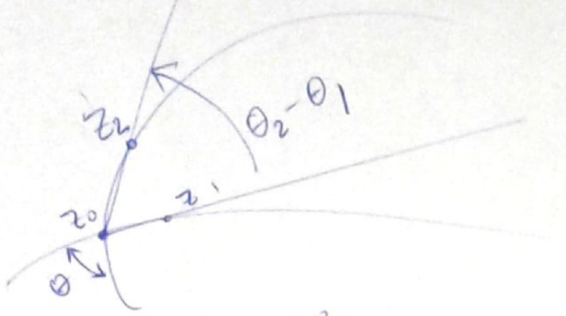
$$\delta w = w - w_0 = f(z) - f(z_0) = (z - z_0) f'(z_0) + \frac{(z-z_0)^2}{2} f''(z_0) + \cdots + \frac{(z-z_0)^n}{n!} f^{(n)}(z_0) + \frac{(z-z_0)^{n+1}}{(n+1)!} f^{(n+1)}(z_0)$$

$$\Rightarrow \delta w = \frac{f^{(n)}(z_0)}{n!} (\delta z)^n + \frac{f^{(n+1)}(z_0)}{(n+1)!} (\delta z)^{n+1} + \cdots$$

thus as $\delta z \to 0$

$$\arg(\delta w) \longrightarrow n \arg(\delta z) + \arg f^{(n)}(z_0)$$

$\Rightarrow \angle^{le}$ bet'n 2 infinitesimal line elements at z_0 is <u>magnified</u> by the factor n.



$z_1 - z_0 = re^{i\theta_1}$

$z_2 - z_0 = re^{i\theta_2}$

$\angle^{le}$ bet^n line segments $= (\theta_2 - \theta_1) \rightarrow$ angle bet^n arcs $= (\theta)$ as $r \to 0$.

$$\theta = \lim_{r \to 0} \arg\left(\frac{z_2 - z_0}{z_1 - z_0}\right)$$

$$\phi = \lim_{r \to 0} \arg\left(\frac{f(z_2) - f(z_0)}{f(z_1) - f(z_0)}\right) = n\theta.$$

<u>eg (HW)</u> Let D be the Δ region ~~bdd~~ bdd by $x, y = 0$ $(x = 0, y = 0)$ & $x + y = 1$. Use $w = f(z) = z^2$ & find the transformed Domain D' in w-plane. Does D have a critical pt? What is it's effect on D'?

* Critical pts are also useful in determining if $w = f(z)$ has an inverse!