q_{ij} and q_{ji} s.t. $q_{ij} > 0$ and $q_{ji} = 0$ or $q_{ij} = 0$ & $q_{ji} > 0$.
(In these cases, the local balance route to investigating stationarity will be futile)!

---

* Following are elaborate illustrations of 2 types of CTMC (continuous stochastic processes) whose stny D^ns do satisfy the local balance condition.

EXAMPLE (1): $\boxed{\text{Birth/Death process}}$

Only two types of jumps are possible
$i \to i+1$ or $i \to i-1$

$q_{i,i+1} = \lambda_i$ (birth rates) ; $i \geq 0$

$q_{i;\,i-1} = \mu_i$ (death rates) ; $i \geq 1$

$$S = \{0, 1, 2, 3, \ldots\}$$

Q) How to find the stny D^n of such a process?

Local balance:- $\pi_i q_{i,i+1} = \pi_{i+1} q_{i+1,i}$

$\Rightarrow \pi_i \lambda_i = \pi_{i+1} \mu_{i+1} \quad \forall i \geq 0$

$\Rightarrow \pi_{i+1} = \dfrac{\lambda_i}{\mu_{i+1}} \pi_i$

$$\xrightarrow{\text{recursively}} \frac{\lambda_i \lambda_{i-1}}{\mu_{i+1}\mu_i}\pi_{i-1}$$

$$\vdots$$

$$= \frac{\lambda_i \cdots \lambda_0}{\mu_{i+1}\cdots\mu_1}\pi_0$$

But we need to ensure the probability normalization condition $\sum_{i\in S}\pi_i = 1$

$$\cancel{\sum_{i=0}^{\infty}\frac{\lambda_i \cdots \lambda_0}{\mu_{i+1}\cdots\mu_1}\pi_0 = 1 \cdot \frac{\lambda_0}{\mu_1}\pi_0 + \sum_{i=1}^{\infty}\frac{\lambda_i \cdots \lambda_0}{\mu_{i+1}\cdots\mu_1}\pi_0 = 1}$$

$i+1=j$
$i=j-1$

Let $i=j-1$

$i=0 \quad i=1$

$$\pi_j = \sum_{j=1}^{\infty}\frac{\lambda_{j-1}\cdots\lambda_0}{\mu_j\cdots\mu_1}\pi_0$$

$$\sum_{i=0}^{\infty}\pi_i = 1$$

$$\Rightarrow \pi_0 + \sum_{i=1}^{\infty}\pi_i = 1$$

$$= \pi_0$$

Now use $\sum_{i=0}^{\infty}\pi_i = 1$

$$\Rightarrow \pi_0 + \sum_{i=1}^{\infty}\pi_i = 1$$

$$= \pi_0 + \sum_{i=1}^{\infty}\frac{\lambda_{i-1}\cdots\lambda_0}{\mu_i\cdots\mu_1}\pi_0 = 1$$

$$\Rightarrow \pi_0\left\{1 + \sum_{i=1}^{\infty}\frac{\lambda_{i-1}\cdots\lambda_0}{\mu_i\cdots\mu_1}\right\} = 1$$

$$\Rightarrow \pi_0 = \frac{1}{1+\sum_{i=1}^{\infty}\frac{\lambda_{i-1}\cdots\lambda_0}{\mu_i\cdots\mu_1}}; \text{ given Denominator } < \infty.$$

Example (2) :- M/M/1 Queue

Kendall's notation

This is one of the basic queueing models in Queueing theory

→ Single server system

→ customers arrive in queue

→ Clients are served by the server on a first-come first served basis

→ upon service completion, the clients depart the system

eg. Single teller bank queue

M / M / 1

Poisson arrivals s.t. inter-arrival time is exponential w/ rate $\lambda > 0$ ("Memoryless) or "Markov

exponential service time w/ rate $\mu > 0$ (again M stands for "Memoryless or "Markov)

No. of servers in the system = 1

Let $X(t)$:= no. of customers in the system at time t; Then ∵ inter-arrival time & service time are exponentially distributed RVs
$\Rightarrow X(t)$ is a CTMC.

$S = \{0, 1, 2, \dots\}$

Indeed, $X(t)$ is a birth/death process
w/ birth rates $\lambda_i = \lambda$; $\forall\, i \geq 0$
& death rates $\mu_i = \mu$; $\forall\, i \geq 1$

Therefore, from example (1); we see that we will require the condition

$$1 + \sum_{i=1}^{\infty} \left(\frac{\lambda}{\mu}\right)^i < \infty$$ (for stationarity)

or $$\sum_{i=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^i < \infty$$

Geometric series
& it converges
$\Leftrightarrow (\lambda/\mu) < 1$

$\Rightarrow$ for a stny D^n to exist; we must have $\lambda < \mu$

i.e. arrival rate < service rate

i.e. $\lambda < \mu$ is the condition for Stability of the queue

Now, we will restrict our analysis only to a stable queue for which $\lambda < \mu$;

from example (1) above.

$$\Pi_0 = \frac{1}{\sum_{i=0}^{\infty} (\lambda/\mu)^i} \underset{\substack{\text{Geom.}\\ \text{series}}}{\overset{\substack{\text{sum of}\\ \infty}}{=}} \left(\frac{1}{1 - \lambda/\mu} \right)^{-1} = 1 - \frac{\lambda}{\mu}$$

Since $\Pi_i = \dfrac{\lambda_{i-1} \cdots \lambda_0}{\mu_i \cdots \mu_1} \Pi_0$

$$= \left(\frac{\lambda}{\mu}\right)^i \Pi_0$$

$$= \left(\frac{\lambda}{\mu}\right)^i \left(1 - \frac{\lambda}{\mu}\right) \quad \forall \ i \geq 1$$

$$\sim \mathrm{Geom}_0\left(1 - \frac{\lambda}{\mu}\right).$$

$\left\{ \text{i.e. no. of failures before } 1^{st} \text{ success w/ } p = (1 - \lambda/\mu) \right\}$

We will stop here for this course ~~as~~ as far as CTMC is concerned but I would encourage you to study the correspondence bet'n local balance eqns & the concept of time-reversibility of CTMC — at your leisure. #.