Definition. (Reduced row echelon form(rref)):-
A matrix is said to be in rref if it satisfies all the following conditions:-

i. If a row has non zero entries, then the first non zero entry is 1, known as the leading 1 or the **Pivot** element of that row.
ii. If a column has a pivot element, then all the other entries in that columns are 0.
iii. If a row contains a pivot element, then each row above it contains a leading 1 further to the left.

Note:-

i. The third condition implies that row of zeros, if any, appear at the bottom of the matrix.
ii. If a matrix satisfied only the conditions (i.) and (iii.) then the matrix is in the Row echelon form (ref).

Examples of ref:- $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 5 & 7 \\ 0 & 1 & 8 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 6 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Examples of rref:- $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & -33 \\ 0 & 1 & 8 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & -9 & 0 \\ 0 & 1 & 6 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

How to convert the matrix into rref/ref:- Using the following elementry row operations we convert a matrix into rref/ref.

i. Divide a row by a non zero scalar.
ii. Subtract a multiple of a row from another row.
iii. Swap two rows.

Example. *Covert the matrix A into ref and rref.*

$$A = \begin{bmatrix} 2 & 6 & 16 \\ 1 & 3 & 8 \\ 1 & 4 & 10 \end{bmatrix}$$

Solution 1.

$$\begin{aligned}
A &= \begin{bmatrix} 2 & 6 & 16 \\ 1 & 3 & 8 \\ 1 & 4 & 10 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & 3 & 8 \\ 1 & 3 & 8 \\ 1 & 4 & 10 \end{bmatrix} && R_1 \mapsto \frac{R_1}{2} \\
&\sim \begin{bmatrix} 1 & 3 & 8 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix} && R_2 \mapsto R_2 - R_1, R_3 \mapsto R_3 - R_1 \\
&\sim \begin{bmatrix} 1 & 3 & 8 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} && R_2 \longleftrightarrow R_3
\end{aligned}$$

This is the ref form of the matrix A.(because second column contain pivot element but other then pivot element in the second column is not zero.)

$$\sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \qquad R_1 \to R_1 - 3R_2$$

this is the rref of matrix A.

Definition. Rank:-
The rank of a matrix A is the number of pivot element in the rref of the matrix A. In other way we can say that rank of matrix is the number of non zeros row in the rref form of the matrix.

so for the matrix $A = \begin{bmatrix} 2 & 6 & 16 \\ 1 & 3 & 8 \\ 1 & 4 & 10 \end{bmatrix}$ the rref is $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

hence the $rank(A) = 2$.

Note:- If the rank of the matrix is equal to number of the columns then the matrix is full rank.
In the previous example the matrix A is not full rank.

Example:- Let $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ then the $rank(I) = 3 =$number of columns.

so I is the full rank matrix.