no need to mult: $-m_{21}a_{11}$ b/c we know $(a_{21} - m_{21}a_{11}) \equiv 0$ (1)

Arithmetic complexity of Gauss Elimination

* eg. $i=1$; m_{21} is mult. by $a_{12}, a_{13}, \ldots, a_{1n}, a_{1\,n+1}$

$$E_1 : \quad a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = a_{1\,n+1}$$

$$E_2 : \quad a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = a_{2\,n+1}$$

$$\vdots$$

$$E_n : \quad a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = a_{n\,n+1}$$

a total of $(n+1-2+1) = (n-1+1) = (n-i+1)$ where $i=1$

* eg. $i=1$ m_{21}, m_{31}, $\vdots$, m_{n1}

So $(n-1)$ div^n to calculate m_{ji}

* There are $\underbrace{(n-i)}$ div^n

Gauss elimination

$$m_{ji} = \frac{a_{ji}}{a_{ii}}$$

↓

Operations to Echelon form

$$E_j - \underbrace{m_{ji}E_i}$$

↓

Backsubstitution.

* for each i there are $(n-i+1)$ mult^n

And for how many such js? $(n-i)$ of them!

Operations are of 2 types

→ Multiplication or division

→ Addition or Subtraction

So there are $(n-i)$ div. & $(n-i)(n-i+1)$ mult. while reducing the system to Echelon form —— (I)

And for similar reasons there will be $(n-i)(n-i+1)$ add^n/sub^n to compute all the requisite $E_j - m_{ji}E_i$ operations. —— (II)

Operations in Backsubstitution :-

$$x_n = \frac{a_{n\,n+1}}{a_{nn}}$$

$$x_i = \frac{a_{i\,n+1} - \sum_{j=i+1}^{n} a_{ij}x_j}{a_{ii}} \quad \text{for } i = n-1, \dots, 1$$

Type of operations	Number of such op^n.
divisions	$1 + (n-1)$
multiplications	$(n-i)$ for each i
Total :	$1+(n-1) + \sum_{i=1}^{n-1}(n-i) = \frac{n^2+n}{2}$

(III) ——

$$x_i = \frac{a_{i\,n+1} - \sum_{j=i+1}^{n} a_{ij}x_j}{a_{ii}}$$

Types of operations	No. of such operations
Add^n	for each i, $(n-i-1)$ b/c there are $(n-i)$ terms in $\sum_{j=i+1}^{n} a_{ij}x_j$
Sub	1 for each i
Total:	$\sum_{i=1}^{n-1}[(n-i-1)+1] = \frac{n^2-n}{2}$ — (IV)

So combining (I), (II), (III) & (IV) there are a total of $\frac{n^3}{3}+n^2-\frac{n}{3}$ mult/div.

& there are a total of $n^3/3 + n^2/2 - 5n/6$ Add^n/Sub^n.

for large n, $n^3/3$ is the dominant term.

& $\therefore$ Total no. of mult/div $\sim O\left(n^3/3\right)$

Total no. of Add^n/Sub $\sim O\left(n^3/3\right)$

in Gauss Elimination!

* for Gauss-Jordan method; for $n \to \infty$ there are a total of $O\left(n^3/2\right)$ mult/div & $O\left(n^3/2\right)$ add^n/sub.

Matrix factorization & reduction in arithmetic complexity.

We are solving $A\vec{x} = \vec{b}$

If $\underbrace{A \equiv LU}_{\text{possible when Gauss elimination can be performed on } A\vec{x}=\vec{b} \text{ w/o row-interchanges}}$;

$$L = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ m_{21} & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ m_{n1} & m_{n2} & \cdots & m_{n\,n-1} & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \cdots & a_{1n}^{(1)} \\ 0 & a_{22}^{(2)} & \ddots & \vdots \\ \vdots & 0 & \ddots & a_{n-1\,n}^{(n-1)} \\ 0 & \cdots & 0 & a_{nn}^{(n)} \end{pmatrix}$$

$$L\underbrace{U\vec{x}} = \vec{b}$$

$$L\vec{y} = \vec{b}$$

So first solve $\cancel{U\vec{x} = \vec{y}}$ $L\vec{y} = \vec{b}$ — (i)

& then solve $U\vec{x} = \vec{y}$ — (ii)

What is the advantage?

$L\vec{y} = \vec{b}$ can be solved in $O(n^2)$ computations

$U\vec{x} = \vec{y}$ can be solved in $O(n^2)$ computations

So complexity of Gauss elimination reduces from $O(n^3/3)$ to $O(2n^2)$.

eg if $n = 100$; $2n^2 = 2 \times 10^4 = 20{,}000$

$$\frac{n^3}{3} = \frac{100 \times 100 \times 100}{3} = \frac{1000000}{3}$$

$$\approx 3{,}33{,}333$$

Red^n in complexity $\approx 94\%$.

Q) Solve the following linear system by LU factoriz" and Gauss elimination applied to the factorized matrices:

$$x_1 + x_2 + 0x_3 + 3x_4 = 4$$
$$2x_1 + x_2 - x_3 + x_4 = 1$$
$$3x_1 - x_2 - x_3 + 2x_4 = -3$$
$$-x_1 + 2x_2 + 3x_3 - x_4 = 4$$

Ans) following sequence of operations reduces the system to Echelon form

$*\ (E_2 - 2E_1) \to (E_2)$

$\#\ (E_3 - 3E_1) \to (E_3)$

$**\ (E_4 - (-1)E_1) \to (E_4)$

$\#\ (E_3 - 4E_2) \to (E_3)$

$**\ (E_4 - (-3)E_2) \to (E_4)$

given →

$$x_1 + x_2 + 0x_3 + 3x_4 = 4$$
$$-x_2 - x_3 - 5x_4 = -7$$
$$3x_3 + 13x_4 = 13$$
$$-13x_4 = -13$$

so $A = \begin{pmatrix} 1 & 1 & 0 & 3 \\ 2 & 1 & -1 & 1 \\ 3 & -1 & -1 & 2 \\ -1 & 2 & 3 & -1 \end{pmatrix}$

$* $ coeff of E_1 to compute E_2

$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & 3 & 13 \\ 0 & 0 & 0 & -13 \end{pmatrix}$

$\#$ Coeffs. to compute E_3

$**$ Coeffs to compute E_4

$= L * U$

So first we need to solve $L\vec{y} = \vec{b}$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 4 \\ -7 \\ 13 \\ -13 \end{pmatrix}$$

fwd. substitution yields $y_1 = 4$

$y_2 = -7 - 2y_1 = -15$

$y_3 = 13 - 3y_1 - 4y_2 = 61$

$y_4 = -13 + y_1 + 3y_2$

$= -9 - 45 = -54$

Now solve

$U\vec{x} = \vec{y}$

$$\Rightarrow \begin{pmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & 3 & 13 \\ 0 & 0 & 0 & -13 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 4 \\ -15 \\ 61 \\ -54 \end{pmatrix}$$

Use back substitution: $x_4 = \frac{54}{13}$; $x_3 = \frac{61 - 13 \times \frac{54}{13}}{3} = \frac{7}{3}$, etc