Review Questions on Series

Q.1) Test the series for convergence or divergence

$$\sum_{n=1}^{\infty} \frac{\sin(1/n)}{\sqrt{n}}$$

<u>Soln:-</u> Let $a_n = \dfrac{\sin(1/n)}{\sqrt{n}} > 0$ for very large n

& $b_n = \dfrac{1}{n\sqrt{n}} > 0$

Then by Limit comparison test,

$$\therefore \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin(1/n)}{(1/n)}$$

$$= \lim_{m\to 0} \frac{\sin m}{m}$$

$$= 1 > 0$$

$\Rightarrow \sum a_n$ converges b/c $\sum b_n$ converges b/c it is convergent p-series w/ $p = 3/2$.

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<u>Recall:-</u>

<u>Limit Comparison test</u>

Let $a_n, b_n > 0 \ \forall n \geq N$ (Integer)

① $\lim_{n\to\infty} \dfrac{a_n}{b_n} = c > 0 \Rightarrow \sum a_n$ & $\sum b_n$ both converge or both diverge.

(II) $\lim_{n\to\infty} a_n/b_n = 0$ and $\sum b_n$ converges $\Rightarrow \sum a_n$ conv.

(III) $\lim_{n\to\infty} a_n/b_n = \infty$ & $\sum b_n$ div $\Rightarrow \sum a_n$ div.

Pg (2)

Q-2) Test the following series for convergence or divergence

$$\sum_{K=1}^{\infty} \frac{K \log K}{(K+1)^3}$$

Soln:- Here we observe terms comprising of log f^n / polynomial f^n which are by themselves continuous for large K and if we consider $\frac{\log x}{x^2} = f(x)$ we see $f(x)$ is decreasing, +ve & continuous $\forall x$ large (surely $\forall x \geq 2$)

$$\int_1^{\infty} \frac{\log x}{x^2} dx = \lim_{\alpha \to \infty} \int_1^{\alpha} \frac{\log x}{x^2} dx$$

$$\overset{IBP}{=} \lim_{\alpha \to \infty} \left. \log x \int \frac{1}{x^2} dx - \int \left(\frac{1}{x} \int \frac{1}{x^2} dx \right) dx \right|_1^{\alpha}$$

$$= \lim_{\alpha \to \infty} \left(-\frac{\log x}{x} + \int \frac{1}{x^2} dx \right)_1^{\alpha}$$

$$= \lim_{\alpha \to \infty} \left(-\frac{\log x}{x} - \frac{1}{x} \right)_1^{\alpha}$$

$$= \lim_{\alpha \to \infty} -\frac{\log \alpha}{\alpha} + \frac{\log 1}{1} - \frac{1}{\alpha} + \frac{1}{1}$$

$$\text{L'hopital's} \; \lim_{\alpha \to \infty} \frac{-1/\alpha}{1} + 1$$

$$= 1$$

$$\Rightarrow \sum_{K=1}^{\infty} \frac{\log K}{K^2} \text{ is convergent}$$

Now

$$\frac{K \log K}{(K+1)^3} < \frac{K \log K}{K^3} = \frac{\log K}{K^2}$$

$\Rightarrow$ By direct comparison

$$\sum_{K=1}^{\infty} \frac{K \log K}{(K+1)^3} < \infty \quad \#$$

Q.3: Test $\sum_{n=1}^{\infty} \frac{(-2)^{2n}}{n^n}$ for conv./div.

Soln:-

Just observing the form of the integrand begs application of the root test.

$$a_n = \frac{((-2)^2)^n}{n^n} = \left(\frac{4}{n}\right)^n \geq 0 \quad \forall n \geq 1$$

$$\lim_{n\to\infty} \sqrt[n]{(4/n)^n} = \lim_{n\to\infty} 4/n = 0 < 1$$

$\Rightarrow \sum a_n$ is convergent by Root test. #

Q.4 Test $\sum_{n=1}^{\infty} (-1)^n \frac{\log n}{\sqrt{n}}$ for conv./divergence.

Soln:- This is an alternating series; so lets try Leibnitz's thm (or Alt. Series test thm)

Need to check :-
i) $a_n > 0$
ii) $a_n \geq a_{n+1}$ (i.e. dec. seq)
iii) $a_n \to 0$

Let $f(x) = \frac{\log x}{\sqrt{x}}$

$$f'(x) = \log x \left(-\tfrac{1}{2}\right) x^{-3/2} + \frac{1}{\sqrt{x}} \frac{1}{x} = \frac{2 - \log x}{2x^{3/2}} < 0$$

when $\log x > 2$ or $x > e^2$.

$\Rightarrow \frac{\log n}{\sqrt{n}}$ is decreasing for $n > e^2$

L'hopitals

Now, $\lim_{n \to \infty} \frac{\log n}{\sqrt{n}} \overset{L}{=} \lim_{n \to \infty} \frac{1/n}{\frac{1}{2}(n^{-1/2})}$

$= \lim_{n \to \infty} \frac{2}{\sqrt{n}} = 0$.

$\Rightarrow \sum a_n$ conv. by Leibniz's th^n.

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Q.5) Test $\sum_{n=1}^{\infty} \frac{n^2-1}{n^2+n}$ for conv/div.

soln:-

$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{n^2-1}{n^2+n}$

$= \lim_{n \to \infty} \frac{1 - 1/n^2}{1 + 1/n}$

$= 1 \neq 0$

$\Rightarrow \sum a_n$ div. by n^{th} term test.

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Q.6)

Determine the n^{th} term of the series & test its conv./div.

$$\frac{1}{2} - \frac{2}{3} \times \frac{1}{2^3} + \frac{3}{4} \cdot \frac{1}{3^3} - \frac{4}{5} \cdot \frac{1}{4^3} + \cdots$$

Soln:- By inspection;

the 1^{st} set of seq are

$$\frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \cdots$$

$(-1)^{n+1} \frac{n}{n+1}$

& the 2^{nd} set of seq are

$$\frac{1}{1^3}, \frac{1}{2^3}, \frac{1}{3^3}, \frac{1}{4^3}, \cdots$$

$\frac{1}{n^3}$

We have i.e. $a_n = (-1)^{n+1} \frac{n}{n+1} \times \frac{1}{n^3}$

$$= (-1)^{n+1} \frac{1}{n^2(n+1)}$$

Now since $|a_n| = \frac{1}{n^2(n+1)} < \frac{1}{n^3}$

& $\sum_{n \geq 1} \frac{1}{n^3}$ is convergent by $p = 3$ series test

We have by Direct comparison,

$\sum_{n \geq 1} |a_n|$ is conv $\xRightarrow{Th^m} \sum_{n \geq 1} a_n = \sum_{n \geq 1} \frac{(-1)^{n+1}}{n^2(n+1)}$ is conv. #

Q.7. Determine the n^{th} term & test the conv / div. of the series

$$\frac{2}{3} - \frac{3}{4}\cdot\frac{1}{2} + \frac{4}{5}\cdot\frac{1}{3} - \frac{5}{6}\cdot\frac{1}{4} + \cdots$$

Soln:-

By inspection,

$$a_n = (-1)^{n+1}\frac{(n+1)}{(n+2)}\frac{1}{n} = (-1)^{n+1} b_n$$

Note $b_n > 0$

$$f(x) = \frac{x+1}{x(x+2)} = \frac{1}{(x+2)} + \frac{1}{x(x+2)}$$

$$f'(x) = -\frac{1}{(x+2)^2} + \left\{ -\frac{1}{(x+2)x^2} - \frac{1}{x}\frac{1}{(x+2)^2} \right\}$$

$$< 0 \quad \forall x > 0.$$

$\therefore b_n \downarrow \quad \forall n \geq 1$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n+1}{n^2+2n} = \lim_{n\to\infty} \frac{1/n + 1/n^2}{1 + 2/n} = 0$$

$\therefore$ By alternating series test $\sum_{n\geq 1} a_n$ is convergent.

But note $|a_n| = \frac{n+2}{(n+1)n}$ & consider $c_n = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{|a_n|}{c_n} = \frac{n+2}{(n+1)n} \times n = \lim_{n\to\infty} \frac{n+2}{n+1} = 1$$ and since $\sum c_n$ div.

$\Rightarrow \sum |a_n|$ is div.

$\therefore \sum a_n$ is conditionally conv.

b/c $\sum_{n\geq 1} \frac{1}{n}$ is harmonic series

$\Rightarrow \sum_{n\geq 1} |a_n|$ also div. by limit comparison test.

thus collecting the results above, we have

$$\left.\begin{array}{l}\sum_{n\geq 1} a_n < \infty \\ \text{but } \sum_{n\geq 1} |a_n| > \infty\end{array}\right\} \Rightarrow \sum_{n\geq 1} a_n$$

is conditionally convergent #.

Q.8) Approximate the sum

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n} = 1 - \frac{1}{4} + \frac{1}{16} - \frac{1}{64} + \cdots$$

upto two decimal places using the method based on the alternating series test. Check your answer by finding the actual sum.

Soln :- We must find the least n

s.t. $\frac{1}{4^n} < 0.005 = \frac{1}{200}$

i.e. $200 < 4^n \Rightarrow n \geq 4$

So if we use $1 - \frac{1}{4} + \frac{1}{16} - \frac{1}{64} = \frac{51}{64}$,

the error will be less than 0.005

$\therefore \frac{51}{64} = 0.796 \Rightarrow$ our approx^n is 0.80

To check, the given series is a geometric series w/ $r = -\frac{1}{4}$

$\therefore$ the ∞-Sum, $S_\infty = \frac{1}{1-(-\frac{1}{4})} = 4/5$
$= 0.8$

$\therefore$ Our approx^n is actually the exact value of the sum.

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Q.9. Estimate the error when $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \dots$ is approximated by its first 10 terms.

Soln:- the error is less than the magnitude of the 1st omitted term i.e. $\frac{1}{11^2} = 0.0083$.

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Q.10.

How many terms must be used to approximate the sum in Q9) correctly to 1 decimal place?

Ans:- We must have $\frac{1}{n^2} < 0.05 = \frac{1}{20}$

$\Rightarrow 20 < n^2$

$\Rightarrow n^2 \geq 5$.

i.e. only $n < 5$ will affect values upto 1 dec. pt.

Note the 5^{th} term $b_5 = \frac{1}{5^2} = \frac{1.00}{25}$

$= 0.04$

If we use $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16}$

$= 0.7986$

≈ 0.8

$b_5 = 0.04$ will not change

$S_{approx} = 0.8$ to one decimal pt.

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Q.11. Determine if $\sum_{n=1}^{\infty} \frac{n^2}{e^n} < \infty$?

Soln. Apply ratio test,

$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(n+1)^2}{e^{n+1}} \cdot \frac{e^n}{n^2} \right|$

$= \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^2 \frac{1}{e} = \frac{1}{e} < 1$

$\Rightarrow$ Series is conv. #.

Q.13. Determine if $\sum_{n=1}^{\infty} \frac{\tan^{-1} n}{n^2+1} < \infty$?

Soln.

Note

$$\frac{\tan^{-1} n}{n^2+1} < \frac{\pi/2}{n^2} \quad \forall n$$

& $\because \pi/2 \sum_{n \geq 1} \frac{1}{n^2} < \infty$ (p = 2 series)

$\Rightarrow$ By direct comparison test; we have

$$\sum_{n \geq 1} \frac{\tan^{-1} n}{n^2+1} < \infty .$$

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Q.13. Find the values of x for which the series

$\log x + (\log x)^2 + (\log x)^3 + \cdots$

converges & express the sum as a f^n of x.

Soln :- This is a geom. series w/ $r = \log x$

$\Rightarrow$ Conv. for $|\log x| < 1$

$\Rightarrow -1 < \log x < 1$

$S = \frac{\log x}{1-\log x}$ $\quad \Rightarrow 1/e < x < e$. #

Q. 14 Does $\sum_{n \geq 1} \frac{n}{n!} < \infty$?

Soln :-

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$= \lim_{n \to \infty} \left| \frac{(n+1)}{(n+1)!} \frac{n!}{n} \right|$$

$$= \lim_{n \to \infty} \left| \frac{1}{n} \right| = 0 < 1$$

$\Rightarrow$ Conv !

Q. 15 Find the radius of convergence & interval of convergence of the series $\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}} = \sum_{n \geq 0} a_n$

Soln :- $a_n = \frac{n(x+2)^n}{3^{n+1}}$

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(n+1)(x+2)^{n+1}}{3^{n+1+1}} \times \frac{3^{n+1}}{n(x+2)^n} \right|$$

$$= \left(1 + \frac{1}{n}\right) \frac{|x+2|}{3} \to \frac{|x+2|}{3} \text{ as } n \to \infty$$

i.e. By ratio test, the series converges if $\frac{|x+2|}{3} < 1$ & div. if $\frac{|x+2|}{3} > 1$

i.e. <u>conv</u> & <u>div</u>

$|x+2| < 3$ $\quad$ $|x+2| > 3$

$\Rightarrow$ R.O.C : $R = 3$.

<u>For interval of conv.</u>

Re-write

$|x+2| < 3$

$\Rightarrow -3 < x+2 < 3$

$\Rightarrow -5 < x < 1$

Now we need to test the series at the end pts. $x = -5, 1$

at $x = -5$; $\sum_{n\geq 0} a_n = \frac{1}{3} \sum_{n\geq 0} (-1)^n n > \infty$

by n^{th} term test.

at $x = 1$; $\sum_{n\geq 0} a_n = \frac{1}{3} \sum_{n=0}^{\infty} n > \infty$.

$\therefore$ the interval of conv. is $(-5, 1)$.

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