Lecture (19)

Argument principle, Rouché's Th^m & Definite Integrals w/ branch points.

WARNING :-

Earlier we have seen

$$\boxed{\sum_{j=1}^{N} \mathrm{Res}(f(z); z_j) = \mathrm{Res}(f(z); \infty)}$$ for

$\{z_1, \dots, z_N\}$ isolated s.p.s of $f(z)$.

Use this formula only when the z_js are "isolated" s.p.s; in case z_js are multi-valued then do not use this formula as in the latter case the singularities are branch pts. (not isolated s.p.s).

Th^m (19.1) Argument-principle

Let $f(z)$ be a meromorphic f^n defined inside & on a Jordan contour C; w/ no zeros/poles on C.

then $$I = \frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = N - P = \frac{1}{2\pi}\left[\arg f(z)\right]_C$$

Where N = no. of zeros of $f(z)$ w/in C
P = no. of poles of $f(z)$ w/in C
$[\arg f(z)]_C$ = change in argument of $f(z)$ over C.

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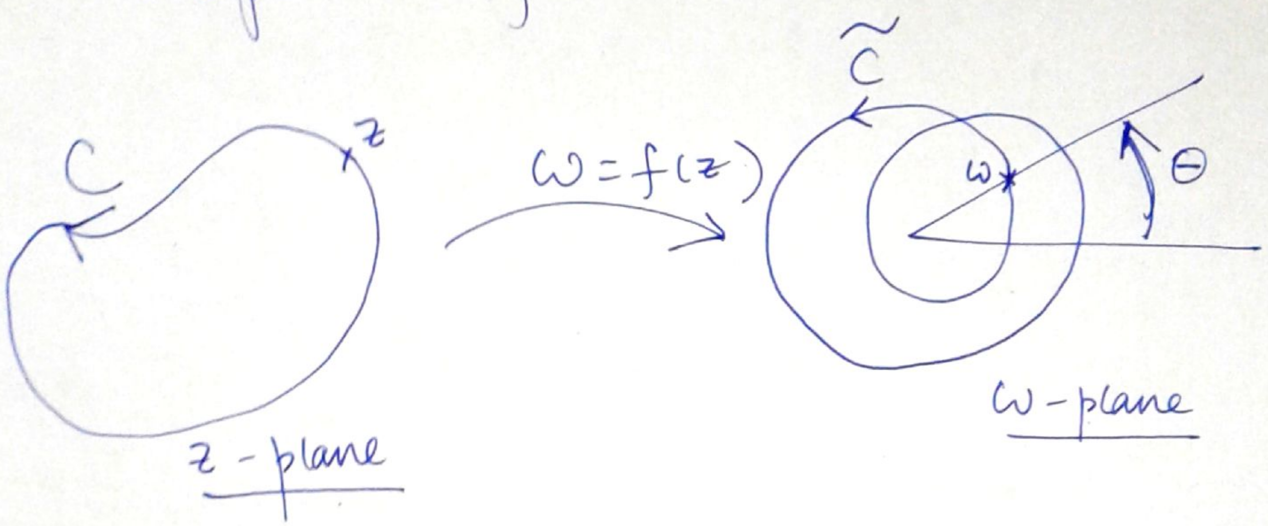
<u>Note</u> $f(z) = |f(z)| e^{i \arg f(z)}$

In the above result, multiple zeros & poles are counted according to their multiplicities of $f(z)$ is as in th^m (19.1).

** $\omega = f(z)$ be s.t. $\omega = |f(z)| e^{i \arg f(z)}$ maps as follows.



then $\frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)} dz = \frac{1}{2\pi i} \oint_{\tilde{C}} \frac{dw}{\omega} = \frac{1}{2\pi i} [\arg \omega]_{\tilde{C}}$

Where $\frac{1}{2\pi i} [\arg \omega]_{\tilde{C}}$ is called the <u>winding number</u> of $\tilde{C}$ about Origin in w-plane.

** Further; if $h(z)$ is analytic inside & on C; then

$$\operatorname{Res}\left(\frac{f'(z)}{f(z)} h(z); z_j\right) = \pm n_j h(z_j);$$

n_j is the order of pole/zero of $f(z)$ at $z = z_j$.

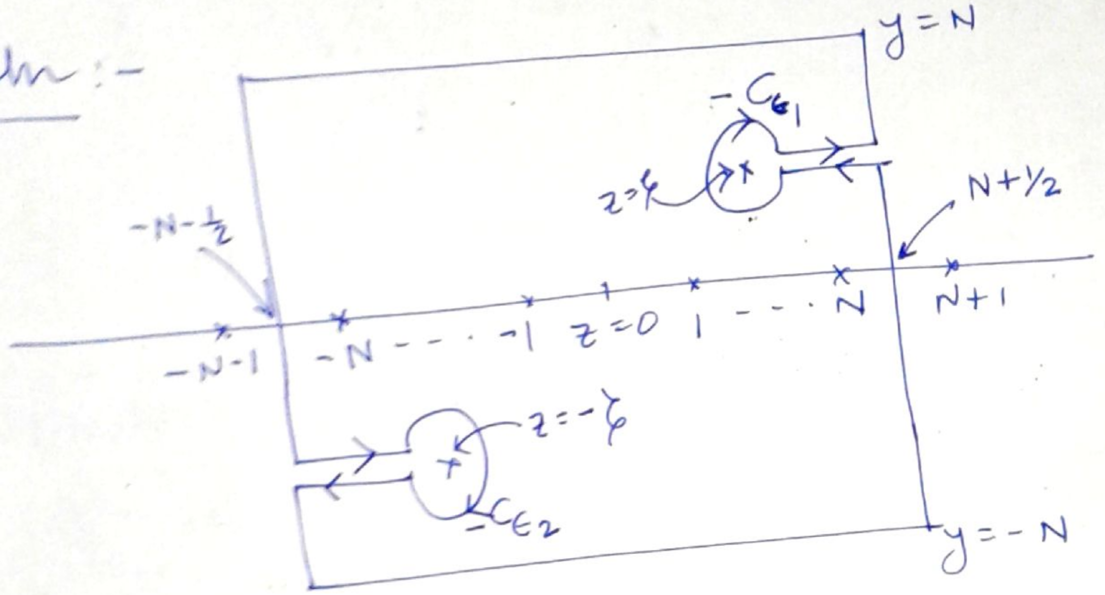
eg (19.1) Consider the following integral

$$I(\xi) = \frac{1}{2\pi i}\oint \frac{\pi \cot \pi z}{(\xi^2 - z^2)}\,dz \;;$$

where C_N is depicted C_N as follows.

Deduce that $\pi \cot \pi\xi = \xi \sum_{n=-\infty}^{\infty} \frac{1}{(\xi^2 - n^2)}$.

Soln :-



Following Th^m (19.1) ! —

$$J = \frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)} h(z)\,dz \;; \quad h(z) \text{ is analytic inside \& on } C;$$

$$= \sum_{i_z=1}^{M_z} n_{iz} h(z_{iz}) - \sum_{i_p=1}^{M_p} n_{ip} h(z_{ip}); \quad f(z) \text{ is as defined in Th}^m (19.1) \quad (1)$$

Here if $f(z) = (z - z_i)^{n_{iz}} g(z)$; $g(z_i) \neq 0$

$\Rightarrow f$ has a zero of order n_{iz}; $g(z)$ is analytic in $B_\epsilon(z_i)$

If $f(z) = \frac{g(z)}{(z - z_i)^{n_{ip}}} \Rightarrow f$ has a pole of order n_{ip} & $g(z)$ is analytic in $B_\epsilon(z_i)$

then in this case;

$$f(z) = \sin \pi z$$

$$h(z) = \frac{1}{\xi^2 - z^2};$$

$z_i = n$ and $n = 0, \pm 1, \ldots, \pm N$.

w/ $n_{iz} = 1$

$n_{ip} = 0$

$$\frac{1}{2\pi i}\left(\oint_{C_{NR}} - \oint_{C_{\epsilon_1}} - \oint_{C_{\epsilon_2}}\right) \frac{\pi \cot \pi z}{\xi^2 - z^2}\, dz$$

from eq(1) $$= \sum_{n=-N}^{N} \frac{1}{\xi^2 - n^2}$$; where C_{NR} denotes the rectangular contour w/o cross-cuts & circles around $z = \pm \xi$.

Taking $N \to \infty$; $\epsilon_i \to 0$, after computing the residues about C_{ϵ_i}

$$\lim_{N\to\infty} \frac{1}{2\pi i} \oint_{C_{NR}} \frac{\pi \cot \pi z}{\xi^2 - z^2}\, dz - \left\{ \left(\frac{\pi \cot \pi z}{-2z}\right)_{z=\xi} + \left(\frac{\pi \cot \pi z}{-2z}\right)_{z=-\xi} \right\}$$

$$= \sum_{n=-\infty}^{\infty} \frac{1}{\xi^2 - n^2}$$

$\because \oint_{C_{NR}} \to 0$ as $N \to \infty$ (H.W.); we have

$$\pi \cot \pi \xi = \xi \sum_{n=-\infty}^{\infty} \frac{1}{\xi^2 - n^2}$$ (Mittag-Leffler expansion of $\pi \cot \pi \xi$). # Pg(4)

Th^m (19.2) (Rouché's Th^m)

Let $f(z)$ and $g(z)$ be analytic on & inside a Jordan contour C.

If $|f(z)| > |g(z)|$ on C; then $f(z)$ and $(f(z)+g(z))$ have the same no. of zeros inside C!

Application (Fundamental Th^m of Algebra).

To prove:-

Every polynomial $P(z) = z^n + a_{n-1}z^{n-1} + \dots + a_0$ has n & only n roots counting multiplicities:-

$P(z_i) = 0$; $i = 1, 2, \dots, n$

Proof:-

$$f(z) = z^n$$

$$g(z) = a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \dots + a_0.$$

for $|z| > 1$; we find

$$|g(z)| \leq |a_{n-1}||z^{n-1}| + |a_{n-2}||z^{n-2}| + \dots + |a_0|$$

$$= |a_{n-1}||z|^{n-1} + |a_{n-2}||z|^{n-2} + \dots + |a_0|$$

$$\leq (|a_{n-1}| + |a_{n-2}| + \dots + |a_0|)|z|^{n-1}$$

Contour C is circle w/ radius $R > 1$;

$|f(z)| = R^n > |g(z)|$ whenever

$R > \max(1, |a_{n-1}|, \dots, |a_0|)$

$\therefore P(z) = f(z) + g(z)$ has the same no of roots as $f(z) = z^n = 0$ which is n.

Moreover, all of the roots of $P(z)$ are contained inside the circle $|z| < R$ by by the above estimate for R

$$|P(z)| = |z^n + g(z)| \geq R^n - |g(z)| > 0$$

& $\therefore$ does not vanish for $|z| \geq R$.

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HW

Q) Show that all the roots of $P(z) = z^8 - 4z^3 + 10$ lie bet'n $1 \leq |z| \leq 2$.

Ostrowski - Hadamard th^m does not apply in the case of analytic continuation of $\sum_{k=0}^{\infty} z^k$ b/c of the following :-

Let us expand $\frac{1}{1-z}$ about $z = ih$;
$0 < h < 1$

$$\frac{1}{1-z} = \frac{1}{(1-ih)-(z-ih)}$$

$$= \left(\frac{1}{1-ih}\right) \frac{1}{1 - \frac{z-ih}{1-ih}} = \left(\frac{1}{1-ih}\right) \sum_{n=0}^{\infty} \frac{(z-ih)^n}{(1-ih)^n}$$

$$= \left(\frac{1}{1-ih}\right) \sum_{n=0}^{\infty} \underbrace{\frac{1}{(1-ih)^n}}_{a_n} (z-ih)^n$$

$$R.O.C., \; R = \frac{1}{\lim_{n\to\infty} \sqrt[n]{\frac{1}{|1-ih|^n}}} = |1-ih|$$

$$= \sqrt{1+h^2} > 1$$

$\therefore$ Ostrowski - Hadamard th^m does not apply.

the open disk of rad $= R$ abt $z = ih$ Contains pts. such as $(1+h)i$ that are outside the unit disk centered at $z = 0$.

Note: It does not include $z = 1$.

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