Lecture (4): Calculus on the complex plane.

Before we discuss calculus on $\mathbb{C}$; let us review eqns. of simple geometrical objects on $\mathbb{C}$

① Unit circle around (0,0)

$$|z| = 1$$

$$\text{or } z = \frac{1}{\bar{z}}$$

② Circle w/ radius r w/ center $z_0 = x_0 + iy_0$

$$|z - z_0| = r$$

③ Eqn. of a straight line $ax + by = c$ on $\mathbb{C}$

$$Re(\lambda z) = 1 \quad ; \quad \lambda = a - ib$$

Why $Re(\lambda z) = 1$

$\Rightarrow Re((a-ib)(x+iy)) = 1$

$\Rightarrow ax + by = 1$ ✓

Generally, speaking eqn. of a straight line on $\mathbb{C}$ is of the form

$$\beta z + \bar{\beta}\bar{z} + r = 0 \; ; \; \beta \in \mathbb{C}, \; r \in \mathbb{R}$$

Why? Let $\beta = a + ib$

then $\beta z + \bar{\beta}\bar{z} + r = (a+ib)(x+iy) + (a-ib)(x-iy) + r$

$ax - by + ax - by + r = 0$

$\Rightarrow ax - by + r/2 = 0$

$\Rightarrow ax - by + r/2 = 0$

Eqn. of a line.

DeMoivre's Thm:-

Let $z = re^{i\theta}$; $n \in \mathbb{I}$

$z^n = r^n e^{in\theta} = r^n(\cos(n\theta) + i\sin(n\theta))$.

(4.1) Limits, Continuity & Complex differentiation

The concepts of limits & continuity are similar to that of real variables.

Let $\omega = f(z)$ be defined for all pts in some neighborhood of $z = z_0$, except possibly for z_0 itself.

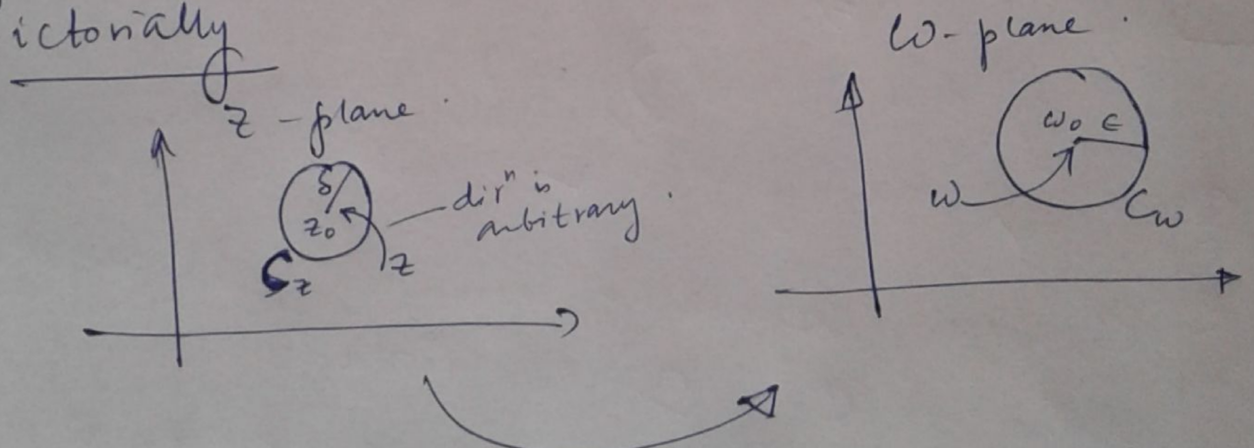
Then $\lim_{z \to z_0} f(z) = \omega_0$ if for $\forall\ \epsilon > 0$ (suff. small)

$\exists$ a $\delta > 0$ s.t.

$$|f(z) - \omega_0| < \epsilon \text{ whenever } 0 < |z - z_0| < \delta .$$

This is true when z_0 is an interior pt. of region R.

Pictorially



map : $\omega = f(z)$

All pts in the interior of $C_z - \{z_0\}$ $\xrightarrow{\text{map}}$ interior of C_ω

eg (4.1)

Show : $\lim_{z \to i} 2\left(\frac{z^2 + iz + 2}{z - i}\right) = 6i$

We must show that given $\epsilon > 0$; $\exists$ a $\delta > 0$

s.t.

$0 < |z - i| < \delta \Rightarrow \left| 2\left(\frac{z^2 + iz + 2}{z - i}\right) - 6i \right|$

$= \left| 2\frac{(z - i)(z + 2i)}{(z - i)} - 6i \right| < \epsilon$

$\because z \neq i$; we ~~have~~ must show

$\left| 2\frac{(z - i)(z + 2i)}{(z - i)} - 6i \right| < \epsilon$

$\Rightarrow 2|z - i| < \epsilon$ ——— (1)

$\therefore$ if we choose $\delta = \epsilon/2$

ineq. (1) will always be true.

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Limit at ∞

$\lim_{z \to \infty} f(z) = w_0 < \infty$ if $\forall \, \epsilon > 0$ (suff. small) $\exists$ a $\delta > 0$

s.t. $|z| > \frac{1}{\delta} \Rightarrow |f(z) - w_0| < \epsilon$.

Properties of limits -

If $f(z) \xrightarrow{z \to z_0} w_0$; $g(z) \xrightarrow{z \to z_0} s_0$

then $(f+g)(z) \xrightarrow{z \to z_0} w_0 + s_0$

$(fg)(z) \xrightarrow{z \to z_0} w_0 s_0$

$(f/g)(z) \xrightarrow{z \to z_0} \frac{w_0}{s_0}$; $s_0 \neq 0$

Continuity of f^ns. (Analogous to real analysis)

$f(z)$ is continuous at $z = z_0$ if.

$$\lim_{z \to z_0} f(z) = f(z_0) \; ; \quad z_0, f(z_0) < \infty .$$

Continuity of f^ns at z_∞.

$\lim_{z \to \infty} f(z) = f(\infty) = w_\infty$ if given $\epsilon > 0$ (suff. small)

$\exists$ a $\delta > 0$ s.t. $|z| > 1/\delta$

$\Rightarrow |f(z) - w_\infty| < \epsilon$.

Chk why?

* Note since $|f(z) - f(z_0)| = |\bar{f}(z) - \bar{f}(z_0)|$

$\Rightarrow$ Continuity of f guarantees continuity of $\bar{f}$.

** If $f(z)$ is continuous at z_0, then

$Re(f(z)) = \frac{f(z) + \bar{f}(z)}{2}$, $Im(f(z)) = \frac{f(z) - \bar{f}(z)}{2i}$

and $|\bar{f}(z)| = f(z)\bar{f}(z)$ are all continuous at $z = z_0$.

Continuous in a region R

We say $f \in R$ is continuous in R i.e. $f \in C(R)$ if it is continuous at every pt. in R.

Uniform continuity (just like real analysis)

Considering continuity in a region R generally requires that $\delta = \delta(\epsilon, z_0)$; $\epsilon > 0$ & $z_0 \in R$. A f^n $f(z)$ is uniformly continuous in R if $\delta = \delta(\epsilon)$ i.e. δ is independent of $z = z_0$.

* Lipschitz continuity $\Rightarrow$ uniform continuity.

* If R is compact & $f \in C(R)$
  $\Rightarrow f$ is uniformly $C(R)$ & bdd!
  Also $\Rightarrow |f(z)|$ attains its max & min. on R.
  this follows from the continuity of $|f(z)|$.

eg. Show that the continuity of $Re(z)$ & $Im(z) \Rightarrow f(z)$ is cont.

$$f(z) = u(x,y) + iv(x,y)$$

We know $\lim\limits_{z \to z_0} f = \lim\limits_{\substack{x \to x_0 \\ y \to y_0}} (u + iv)$

$= u(x_0, y_0) + i v(x_0, y_0)$
$= f(z_0)$

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Derivatives

$$f'(z_0) = \lim_{\Delta z \to 0} \left(\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \right)$$ if R.H.S. limit exists.

$$= \lim_{z \to z_0} \left(\frac{f(z) - f(z_0)}{z - z_0} \right)$$

* A continuous f^n need not be differentiable. Indeed it turns out diff. f^ns possess many spl. properties. (Nxt chp.!).

eg. $f(z) = \bar{z}$; we showed earlier that $f = \bar{z}$ is continuous. We will see later that $f = \bar{z}$ is not differentiable.

* Diff. complex f^ns. are called <u>Analytic f^ns</u>.

If f & g have derivates, then

$$(f \pm g)' = f' \pm g',$$
$$(fg)' = f'g + fg',$$
$$(f/g)' = \frac{f'g - fg'}{g^2} \; ; \; g \neq 0.$$

& $(f(g(z)))' = f'(g(z))\, g'(z)$

like in reals

Derivatives of Elementary f^ns

$$\frac{d}{dz} z^n = n z^{n-1} \; ; \; n \in \underline{I}$$

$$\frac{d}{dz} c = 0 \; ; \; c \text{ const} .$$

$$\frac{de^z}{dz} = e^z$$

$$\frac{d \sin z}{dz} = \cos z \;,\; \frac{d}{dz} \cos z = - \sin z .$$

$$\frac{d}{dz} \sinh z = \cosh z \;,\; \frac{d}{dz} \cosh z = \sinh z .$$

In tutorial, we will see an elementary application of ODEs.

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—— End of basic review of complex no.s & f^ns ——

(the real fun will commence now! Well, from next lecture) . :)