Gauss Elimination for solving systems of linear eqs. (1)

We are solving:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_n \end{aligned}$$

Procedure:

(I) Find an eqn. in which x_1 appears and, if necessary, interchange this eqn. w/ the 1st eqn. Thus we can assume that x_1 appears in eqn. 1.

(II) Multiply eqn. 1 by a suitable non-zero scalar in such a way as to make the coefficient of x_1 equal to 1.

(III) Subtract suitable multiples of eqn. 1 from eqs. 2 through m in order to eliminate x_1 from these eqs.

(IV) Inspect eqs. 2 through m and find the first eq. which involves one of the unknowns $x_2, \ldots, x_n$, say x_{i_2}. By interchanging eqs. once again, we can suppose that x_{i_2} appears in eq. 2.

some scalars

$$
\begin{aligned}
x_{i_1} + {*}x_{i_2} + \cdots + {*}x_n &= * \quad \text{---- (1)}\\
x_{i_2} + \cdots + {*}\,x_n &= * \quad \text{---- (2)}\\
&\;\vdots\\
x_{i_r} + \cdots + {*}\,x_n &= *\\
0 &= *\\
&\;\vdots\\
0 &= * \quad \text{---- (m)}
\end{aligned}
$$

(v) Multiply eq 2. by a suitable non zero scalar to make the coefficient of x_{i_2} equal to 1.

(vi) Subtract multiples of eqn. 2 from eqns. 3 through m to eliminate x_{i_2} from these eqns.

(vii) Examine eqs. 3 through m and find the first one that involves an unknown other than x_1 and x_{i_2}, say x_{i_3}. Interchange eqs. so that x_{i_3} appears in eq. 3.

this elimination procedure continues in this manner producing the so called <u>pivotal unknowns</u> $x_1 = x_{i_1}, x_{i_2}, \ldots, x_{i_r}$ until we reach a linear system in which no further unknowns occur in the eqs. beyond the r^{th} eq.

A linear system of this sort is said to be in <u>echelon form</u>

the i_j are integers which satisfy

$$1 = i_1 < i_2 < \cdots < i_r \leq n$$

After arriving at Echelon form, use <u>back-substitutⁿ</u> to solve for the unknowns $x_1, x_2, \ldots, x_n$.

Q) What can be said about the solution(s) of the linear system by inspecting the Echelon form?

Ans) <u>Theorem</u> :

there exists atleast one solution!

(I) A linear system is consistent if and only if all the entries on the right hand sides of those eqs. in echelon form which contain no unknowns are zero.

(II) If the system is consistent, the non-pivotal unknowns can be given arbitrary values; the general solution is then obtained by using back substitution to solve for the pivotal unknowns.

(III) The system has a unique soln. if and only if all the unknowns are pivotal.

Matrix form of Gauss Elimination.

the elementary row operations to a matrix equivalent to the operations we performed while implementing Gauss elimination to the linear system are as follows :-

(I) $R_i \longleftrightarrow R_j$

(II) $R_i : R_i + c R_j$

(III) $R_i : c R_i$

c is scalar const.

the matrix in row-echelon form will have a descending staircase structure ⟶

$$\begin{pmatrix} 0 \cdots 0 & 1 & * & \cdots & * & \cdots & \cdots & * \\ 0 \cdots 0 & 0 & 0 & \cdots & 0 & 1 & * \cdots & * \\ 0 \cdots 0 & \cdots & \cdots & \cdots & 0 & 0 & 0 & 1 * \cdots * \\ 0 \cdots 0 & \cdots & \cdots & 0 & \cdots & \cdots & 0 & \cdots * \\ 0 \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \cdots * \\ 0 \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \cdots * \\ 0 \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \cdots * \end{pmatrix}$$

eg. of matrix in row-echelon form.

$$\begin{pmatrix} 1 & 3 & 3 & 2 & 1 \\ 2 & 6 & 9 & 5 & 5 \\ -1 & -3 & 3 & 0 & 5 \end{pmatrix}$$

$R_2 : R_2 - 2R_1$ and $R_3 : R_3 + R_1$

$$\begin{pmatrix} 1 & 3 & 3 & 2 & 1 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & 6 & 2 & 6 \end{pmatrix}$$

$R_2 : \frac{1}{3} R_2$ and $R_3 : R_3 - 6R_2$

$$\begin{pmatrix} 1 & 3 & 3 & 2 & 1 \\ 0 & 0 & 1 & 1/3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

This matrix is in row-echelon form!

Consider Q 3 (or case 3) from prev. lecture
i.e. Lec. set 2.9

$$\begin{aligned} x_1 + 3x_2 + 3x_3 + 2x_4 &= 1 \\ 2x_1 + 6x_2 + 9x_3 + 5x_4 &= 5 \\ -x_1 - 3x_2 + 3x_3 &= 5 \end{aligned}$$

can be written as
$AX = B$

the augmented matrix is

$$M = [A \mid B] = \left(\begin{array}{cccc|c} 1 & 3 & 3 & 2 & 1 \\ 2 & 6 & 9 & 5 & 5 \\ -1 & -3 & 3 & 0 & 5 \end{array}\right)$$

In row-echelon form this becomes

$$\left(\begin{array}{cccc|c} 1 & 3 & 3 & 2 & 1 \\ 0 & 0 & 1 & 1/3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

non-pivotal entries!

B/c this is 0 (i.e. 0=0) this system is consistent.

give them arbitrary values $x_4 = c$, $x_2 = d$

$$\equiv \begin{cases} x_1 + 3x_2 + 3x_3 + 2x_4 = 1 \\ x_3 + \frac{1}{3}x_4 = 1 \\ 0 = 0 \end{cases}$$

the pivotal entries x_1 and x_3 can be found by back substitution

$$x_1 = -2 - c - 3d$$

$$x_3 = 1 - c/3$$

Reading Assignment!

Gauss Elimination Algorithm.

→ refer pg. 350–351 of textbook by Burden & Faires (8th edn.)

→ also refer wikipedia page on Gauss Elimination

→ also refer Lab manual of this Course!