Solving systems of linear ODEs w/ complex eigenvalues

We will develop the theory here for a 2×2 system. The generalization to an $n \times n$ system will follow naturally!

$$\vec{x}\,' = A_{2\times 2}\,\vec{x}$$

say has evs

$$\lambda_{1,2} = \alpha \pm i\beta$$

Corresponding EVs will be $\vec{v}_1, \vec{v}_2 = \vec{p} \pm i\vec{q}$

Complex λs and EVs always appear in complex conjugate pairs!

So the full soln. can be written as

$$\vec{x}(t) = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2$$

But since λ_s and $\vec{v}_s$ are complex, we must break up the solution space into real and imaginary parts to investigate the trajectories on the phase plane!

So let's re-write $\vec{x}(t)$ as

$$\vec{x}(t) = \vec{x}_{re}(t) + i\,\vec{x}_{im}(t)$$

How do we do this ?

$$\vec{x}(t) = c_1 e^{(\alpha+i\beta)t}(\vec{p}+i\vec{q}) + c_2 e^{(\alpha-i\beta)t}(\vec{p}-i\vec{q})$$

$$= c_1 e^{\alpha t} \underbrace{e^{i\beta t}}_{\substack{\downarrow \text{ Euler's identity} \\ \cos\beta t + i\sin\beta t}} (\vec{p}+i\vec{q}) + c_2 e^{\alpha t} \underbrace{e^{-i\beta t}}_{\substack{\downarrow \\ \cos\beta t - i\sin\beta t}} (\vec{p}-i\vec{q})$$

$$= \underbrace{c_1 e^{\alpha t}(\cos\beta t\,\vec{p} - \sin\beta t\,\vec{q})}_{\vec{x}_{re}(t)} + \underbrace{c_2 i\, e^{\alpha t}(\sin\beta t\,\vec{p} + \cos\beta t\,\vec{q})}_{\vec{x}_{im}(t)}$$

$(c_2 i)$ is a const. b/c $i = \sqrt{-1}$ is a const.

$$\therefore \vec{x}(t) = c_1 \vec{x}_{re}(t) + c_2 \vec{x}_{im}(t)$$

Question :- Are $\vec{x}_{re}(t)$ and $\vec{x}_{im}(t)$ linearly independent solns. ?

Ans :- Lets plug in $\vec{x}(t) = \vec{x}_{re}(t) + i\,\vec{x}_{im}(t)$ in $\vec{x}' = A\vec{x}$

$$\vec{x}' = A\vec{x} = A\left(\vec{x}_{re} + i\,\vec{x}_{im}\right)$$

$$\Updownarrow$$

$$\vec{x}' = \vec{x}_{re}' + i\,\vec{x}_{im}' = A\left(\vec{x}_{re} + i\,\vec{x}_{im}\right)$$

Now comparing real & imaginary parts of L.H.S. and R.H.S. :-

$$\vec{x}_{re}'(t) = A\vec{x}_{re}(t) \quad \text{and} \quad \vec{x}_{im}'(t) = A\vec{x}_{im}(t)$$

Each of real & imag. parts of $\vec{x}(t)$ satisfy the ODE!

And since a 2x2 system $\vec{x}' = A\vec{x}$ has 2 linearly independent soln; $\vec{x}_{re}$ and $\vec{x}_{im}$ suffice!!

Recall the fundamental matrix $X(t) = \begin{pmatrix} | & | \\ \vec{x}_1 & \vec{x}_2 \\ | & | \end{pmatrix}$ is NOT unique!!

Nice thing: $\vec{x}_{re}(t)$ and $\vec{x}_{im}(t)$ can be studied together on the ph-plane!

eg 1) Solve $\vec{x}' = A\vec{x} = \begin{pmatrix} 6 & -1 \\ 5 & 4 \end{pmatrix}\vec{x}$

Soln:- $\begin{pmatrix} 6 & -1 \\ 5 & 4 \end{pmatrix}$ have evs. $\lambda_{1,2} = 5 \pm 2i$

EVs $\vec{v}_{1,2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \pm i \begin{pmatrix} 0 \\ -2 \end{pmatrix}$

$\therefore$ the general soln. is

$$\vec{x}(t) = c_1 \vec{x}_{re}(t) + c_2 \vec{x}_{im}(t)$$

$$= e^{5t}\left\{ c_1 \begin{pmatrix} \cos 2t \\ \cos 2t + 2\sin 2t \end{pmatrix} + c_2 \begin{pmatrix} \sin 2t \\ \sin 2t - 2\cos 2t \end{pmatrix} \right\}$$

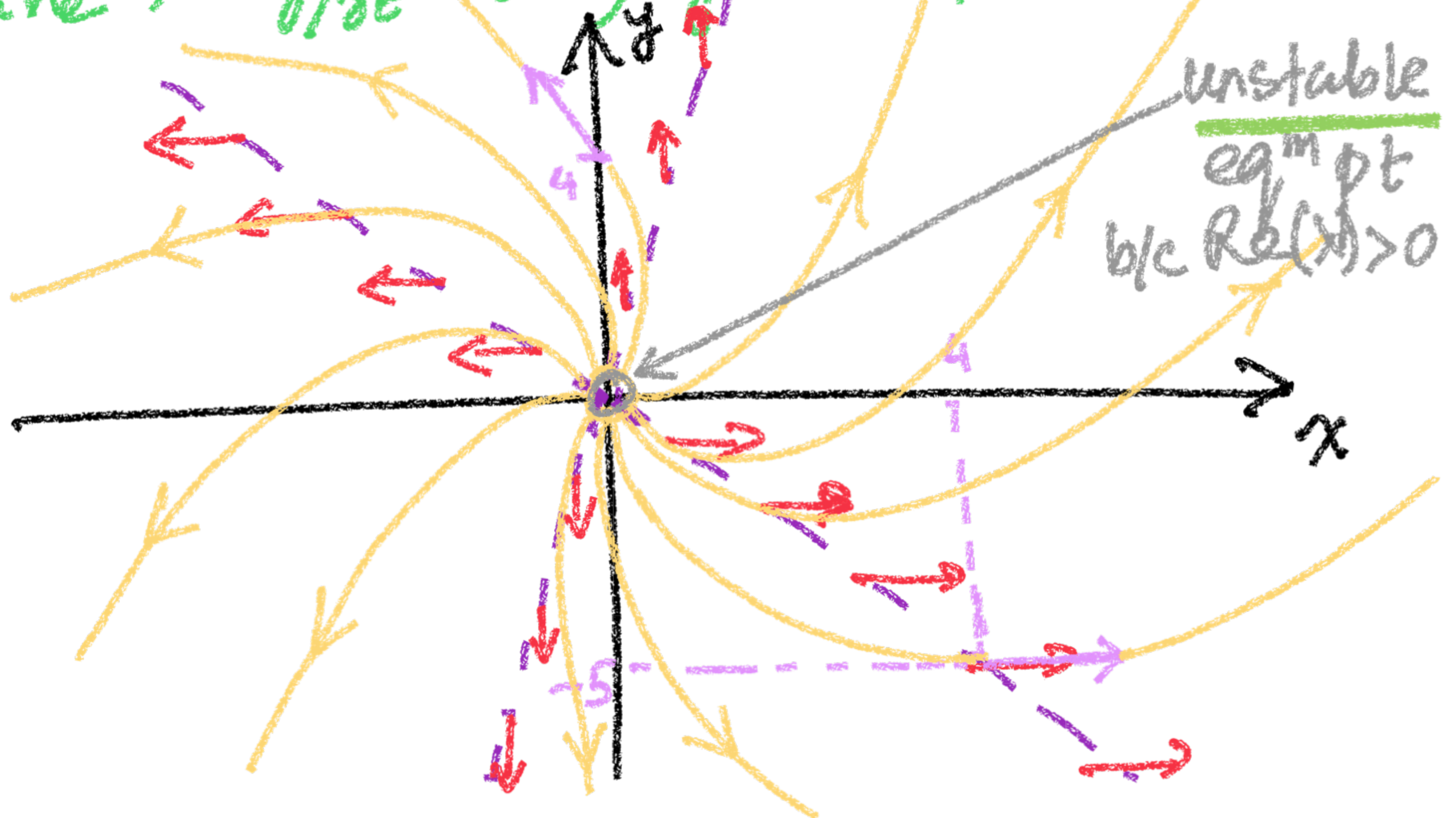
where c_1 & c_2 are real constants

How do we draw the phase portrait?

$$x' = \begin{pmatrix} \partial x/\partial t \\ \partial y/\partial t \end{pmatrix} = \begin{pmatrix} 6 & -1 \\ 5 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

v - nullcline : $\partial x/\partial t = 0 \Rightarrow y = 6x$

h - nullcline : $\partial y/\partial t = 0 \Rightarrow y = -5x/4$



(x,y)	$\frac{dx}{dt}$	$\frac{dy}{dt}$
$(0,4)$	-4 $\leftarrow$	16 $\uparrow$
$(4,-5)$	29 $\rightarrow$	0

eg 2) $\vec{x}' = A\vec{x} = \begin{pmatrix} 4 & -5 \\ 5 & -4 \end{pmatrix}\vec{x}$

Soln :- Lets find the evs. of A.

$|A-\lambda I| = 0 \Rightarrow \lambda^2+9=0 \Rightarrow \lambda_{1,2} = \pm 3i$

EVs :- $\vec{v}_{1,2} = \begin{pmatrix} 5 \\ 4 \mp 3i \end{pmatrix} = \underbrace{\begin{pmatrix} 5 \\ 4 \end{pmatrix}}_{p} \pm i \underbrace{\begin{pmatrix} 0 \\ -3 \end{pmatrix}}_{q}$

$\vec{x}_{re}(t) = \cos 3t \begin{pmatrix} 5 \\ 4 \end{pmatrix} - \sin 3t \begin{pmatrix} 0 \\ -3 \end{pmatrix}$

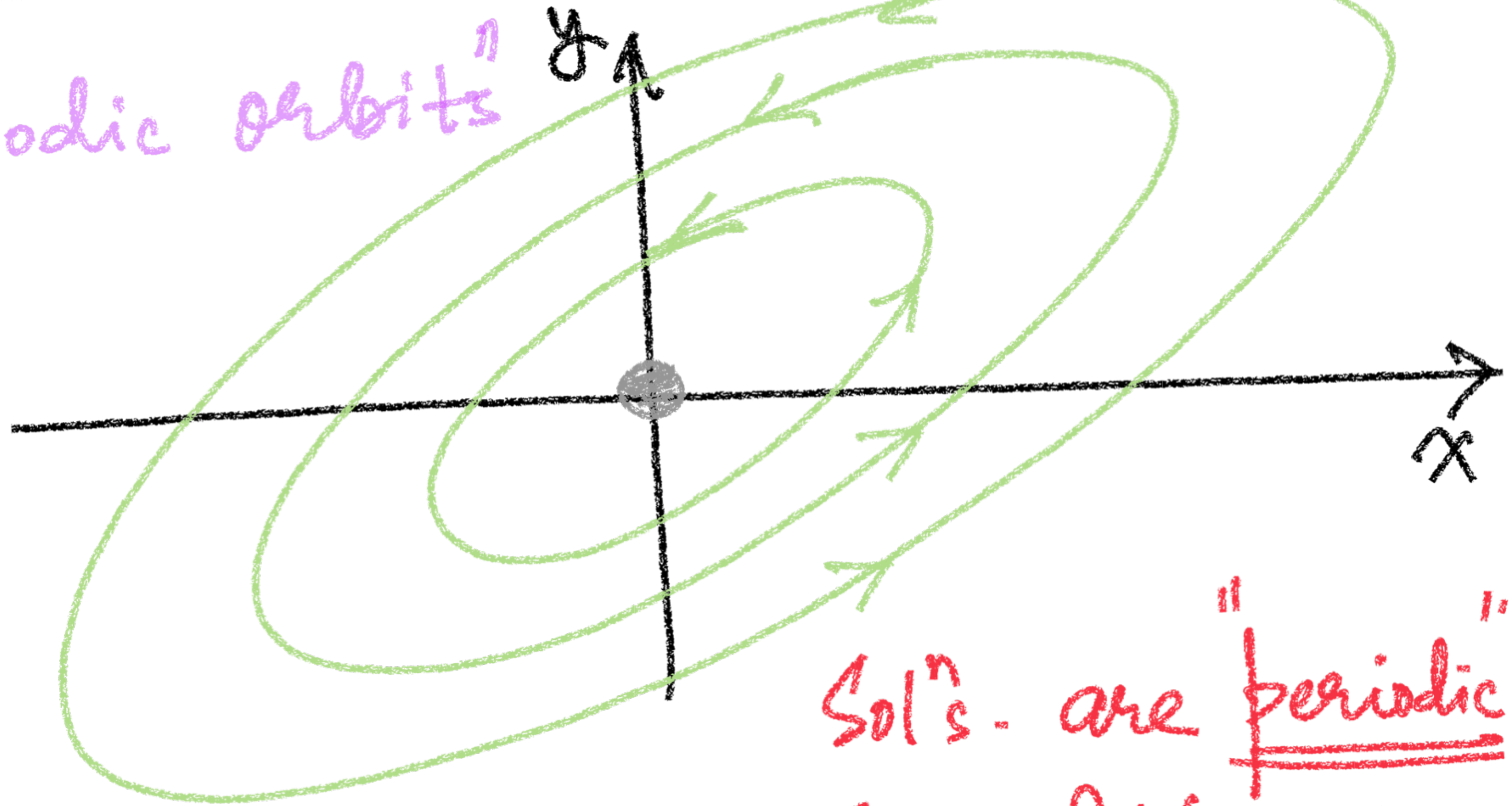
$\vec{x}_{im}(t) = \sin 3t \begin{pmatrix} 5 \\ 4 \end{pmatrix} + \cos 3t \begin{pmatrix} 0 \\ -3 \end{pmatrix}$

General soln. $\vec{x}(t) = c_1 \vec{x}_{re}(t) + c_2 \vec{x}_{im}(t)$

$= c_1 \begin{pmatrix} 5\cos 3t \\ 4\cos 3t + 3\sin 3t \end{pmatrix} + c_2 \begin{pmatrix} 5\sin 3t \\ 4\sin 3t - 3\cos 3t \end{pmatrix}$

How about the phase portrait?

"Periodic orbits"



Sol^{n}s. are "<u>periodic</u>"

when EVs are <u>purely</u> imaginary!

Reading assignments

→ Unstable eq^n.

→ asymptotically eq^m.

→ stable eq^m.

* Refer pg. 378, sec. 6.3

Interpretation of solns.

Real sol^{n}s. from non-real eVs.

$$\vec{x}' = A\vec{x} \longrightarrow \begin{array}{l} \lambda_{1,2} = \alpha \pm i\beta \\ \vec{v}_{1,2} = \vec{p} + i\vec{q} \end{array}$$

To understand this refer to the phase portrait of eg 1 of this lecture!!

Solns. are of the form:

$$\begin{pmatrix} \vec{x}_{re} \\ \vec{x}_{im} \end{pmatrix} = \underbrace{e^{\alpha t}}_{\substack{\text{expansion} \\ \text{(or contraction)}}} \underbrace{\begin{pmatrix} \cos\beta t & -\sin\beta t \\ \sin\beta t & \cos\beta t \end{pmatrix}}_{\substack{\text{Rotation} \\ \beta > 0 \Leftrightarrow \text{counter-clockwise}}} \underbrace{\begin{pmatrix} \vec{p} \\ \vec{q} \end{pmatrix}}_{\text{tilt \& shape}}$$

COMING SOON ON JAN 31

Final ~~Lecture~~ of this sem.

* Stability & Linear classif^n.
* How to find $\vec{x}_p$ for systems of linear ODEs w/ non-homogeneous forcing f^ns.