Convergence Results for General Iteration Methods

Theorem

For any $\mathbf{x}^{(0)} \in \mathbb{R}^n$, the sequence $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ defined by

$$\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c}, \quad \text{for each } k \geq 1$$

converges to the unique solution of

$$\mathbf{x} = T\mathbf{x} + \mathbf{c}$$

if and only if $\rho(T) < 1$.

Convergence of the Jacobi & Gauss-Seidel Methods

Using the Matrix Formulations

We have seen that the Jacobi and Gauss-Seidel iterative techniques can be written

$$\begin{aligned} \mathbf{x}^{(k)} &= T_j \mathbf{x}^{(k-1)} + \mathbf{c}_j \quad \text{and} \\ \mathbf{x}^{(k)} &= T_g \mathbf{x}^{(k-1)} + \mathbf{c}_g \end{aligned}$$

using the matrices

$$T_j = D^{-1}(L+U) \quad \text{and} \quad T_g = (D-L)^{-1}U$$

respectively. If $\rho(T_j)$ or $\rho(T_g)$ is less than 1, then the corresponding sequence $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ will converge to the solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$.

Convergence of the Jacobi & Gauss-Seidel Methods

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This implies that

$$D\mathbf{x} = (L+U)\mathbf{x} + \mathbf{b} \quad \text{and} \quad (D-L-U)\mathbf{x} = \mathbf{b}$$

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Since $D-L-U=A$, the solution $\mathbf{x}$ satisfies $A\mathbf{x}=\mathbf{b}$.

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Theorem

If A is strictly diagonally dominant, then for any choice of $\mathbf{x}^{(0)}$, both the Jacobi and Gauss-Seidel methods give sequences $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ that converge to the unique solution of $A\mathbf{x} = \mathbf{b}$.

Convergence of the Jacobi & Gauss-Seidel Methods

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- No general results exist to tell which of the two techniques, Jacobi or Gauss-Seidel, will be most successful for an arbitrary linear system.
- In special cases, however, the answer is known, as is demonstrated in the following theorem.

Convergence of the Jacobi & Gauss-Seidel Methods

(Stein-Rosenberg) Theorem

If $a_{ij} \leq 0$, for each $i \neq j$ and $a_{ii} > 0$, for each $i = 1, 2, \ldots, n$, then one and only one of the following statements holds:

(i) $0 \leq \rho(T_g) < \rho(T_j) < 1$

(ii) $1 < \rho(T_j) < \rho(T_g)$

(iii) $\rho(T_j) = \rho(T_g) = 0$

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For the proof of this result, see pp. 120–127. of

Young, D. M., Iterative solution of large linear systems, Academic Press, New York, 1971, 570 pp.

Convergence of the Jacobi & Gauss-Seidel Methods

Two Comments on the Thoerem

- For the special case described in the theorem, we see from part (i), namely

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- Part (ii), namely

$$1 < \rho(T_j) < \rho(T_g)$$

  indicates that when one method diverges then both diverge, and the divergence is more pronounced for the Gauss-Seidel method.