Tutorial 2: Calculus on $\mathbb{C}$, Cauchy-Riemann conditions, & analyticity

PART I: Basic calculus on $\mathbb{C}$

1. Use the definition of limit of a complex function to find:
(i) $\lim_{z\to 0}\frac{\bar{z}^2}{z}$ (ii) $\lim_{z\to(1-i)} x+i(2x+y)$, where $z=x+iy$.

2. If $z\neq 0$, use the definition of derivative to show that $(z^{-1})'=-z^{-2}$. Then deduce the formula for $(z^n)'$ using chain rule. Here n is a positive integer.

3. Evaluate the following limits:
(i) $\lim_{z\to i}(z+\frac{1}{z})$ (ii) $\lim_{z\to i}\sinh z$ (iii) $\lim_{z\to 0}\frac{\sin z}{z}$ (iv) $\lim_{z\to\infty}\frac{\sin `z}{z}$

4. Is $f(z)=z^{-2}$ uniformly continuous in $\frac{1}{2}<\mathcal{R}e(z)<1$? Is it so in $0<\mathcal{R}e(z)<\frac{1}{2}$?

PART II: Cauchy-Riemann equations & analyticity

5. Identify the region in which the following functions are differentiable. Then, find $f'(z)$.
(i) $f=\sin z$ (ii) $f=\tan z$ (iii) $f=z\mathcal{R}e(z)$ (iv) $f=x^2+iy^2$

6. Identify the region of analyticity of the following functions. Identify singular points if any.
(i) $\tan z$ (ii) $e^{\sin z}$ (iii) $e^{\frac{1}{z-1}}$ (iv) $e^{\bar{z}}$ (v) $\cos x\cosh y - i\sin x\sinh y$

7. Consider the following complex potential
$$\Omega(z)=-\frac{k}{2\pi z},\qquad k\in\mathbb{R},$$
which is referred to a *doublet*. Calculate the corresponding velocity potential, stream function, and velocity field. Sketch the streamlines.

8. Discuss the flow represented by $\Omega(z)=\log z$. Calculate the corresponding velocity potential, stream function, and velocity field. Sketch the streamlines.

9. (*Constructing harmonic functions*) Let $f(z)=u+iv$ be analytic in the open region R. Assume $u^2+v^2\neq 0$ in R. Show that $\frac{uu_x+vv_x}{u^2+v^2}$ is harmonic in R.
Further, if $w(z)$ is analytic, then is $\frac{w'}{w}$ also analytic?

10. If f is an entire function such that $f(0)=f'(0)=0$ and $Im\big(f'(z)\big)=6xy-2x$, then find $f(1)$.

PART III: Multivalued functions

11. Find the branch points of the following functions and find the cut plane where the functions become single valued.
(i) $\frac{1}{\sqrt{z-1}}$ (ii) $2\log z^2$ (iii) $z^{\sqrt{2}}$ (iv) $z^{\frac{1}{3}}(1-z)^{\frac{2}{3}}$

12. Identify the branch structure (branch points, branch cuts, etc.) of $w=cos^{-1}z$. Then deduce the derivative $\frac{d}{dz}\cos^{-1}z$ on the cut plane.

13. Deduce the identity $\tanh^{-1}z=\frac{1}{2}\log\frac{1+z}{1-z}$. Use it to find $\frac{d}{dz}\tanh^{-1}z$.

14. Let α be a real number. Is the set of all values of the multivalued function $\log z^{\alpha}$ the same as that of $\alpha\log z$?

15. Discuss the branch structure of $\log\{z-\sqrt{z^2+1}\}$ and find its region of analyticity.