We will study more about the vector space of $m \times n$ matrices over the reals and their mathematical utilities! In fact much of this course is a study about matrices and their applications in engineering.

Vector space of $m \times n$ matrices over the reals, $\mathbb{M}_{m\times n}(\mathbb{R})$ and associated vector spaces

Unfortunately, no one can be told what The Matrix is. You'll have to see it for yourself. ***~Morpheus.***

1. So what are matrices?

Ans: Matrices are convenient arrangement of numbers in rows and columns lending a compact structure that are amenable to mathematical laws (laws or rules of matrix algebra) that are a consequence of matrix operations like addition, multiplication, transpose, inverse, etc:

$$A + B = B + A \text{ (commutative law of addition)}$$

$$(A + B) + C = A + (B + C) \text{ (associative law of addition)}$$

$$A + 0 = A$$

$$(AB)C = A(BC) \text{ (associative law of multiplication)}$$

$$AI = A = IA$$

$$A(B + C) = AB + AC \text{ (distributive law)}$$

$$(A + B)C = AC + BC \text{ (distributive law)}$$

$$A - B = A + (-1)B$$

$$(cd)A = c(dA)$$

$$c(AB) = (cA)B = A(cB)$$

$$c(A + B) = cA + cB$$

$$(c + d)A = cA + dA$$

$$(A + B)^T = A^T + B^T$$

$$(AB)^T = B^T A^T$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdot & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & \cdot & a_{2n} \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & \cdot \\ a_{m1} & a_{m2} & \cdot & \cdot & \cdot & a_{mn} \end{pmatrix}$$

$$A^T = \begin{pmatrix} a_{11} & a_{21} & \cdot & \cdot & \cdot & a_{m1} \\ a_{12} & a_{22} & \cdot & \cdot & \cdot & a_{m2} \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & \cdot \\ a_{1n} & a_{2n} & \cdot & \cdot & \cdot & a_{mn} \end{pmatrix}$$

Here, $A, B, C \in \mathbb{M}_{m\times n}(\mathbb{R}), \quad a_{ij} \in \mathbb{R}, \quad c, d \in \mathbb{F}$ where $\mathbb{F} \equiv \mathbb{R}$ in our discussion in this chapter.

2. The rules of matrix algebra guarantee that $\mathbb{M}_{m\times n}(\mathbb{R})$ is a vector space!

3. Special cases: i) When $n = 1$ in $\mathbb{M}_{m\times n}(\mathbb{R})$, we recover the familiar Euclidean space $\mathbb{R}^m$, and ii) when $m = 1$, we recover the vector space $\mathbb{R}_n$ of all real $n-$row vectors.[3]

4. A practical application of matrices: A system of linear equations can be expressed in matrix form and the entire mathematical machinery of matrices can be unleashed to find and analyse the solution(s) of the said system of linear equations.

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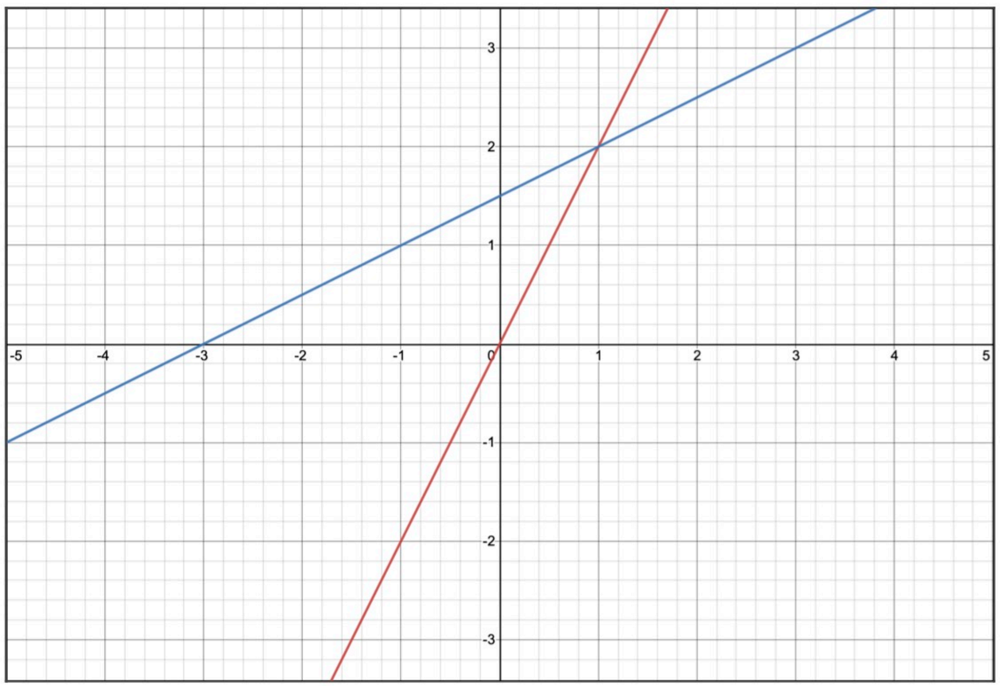
[3] *There is a seamless hierarchy of what are known as tensors in mathematical parlance, the most simplest tensor being scalars (tensors of rank 0), the next in the hierarchy are vectors (tensors of rank 1), followed by matrices (tensors of rank 2), etc. It must be noted that not all matrices are tensors, but all tensors of rank 2 are definitely matrices!*

eg. 4.1: Consider the two set of linear equations:

$$\begin{aligned} 2x - y &= 0 \\ -x + 2y &= 3 \end{aligned} \qquad \text{.....................................} \text{ (i)}$$

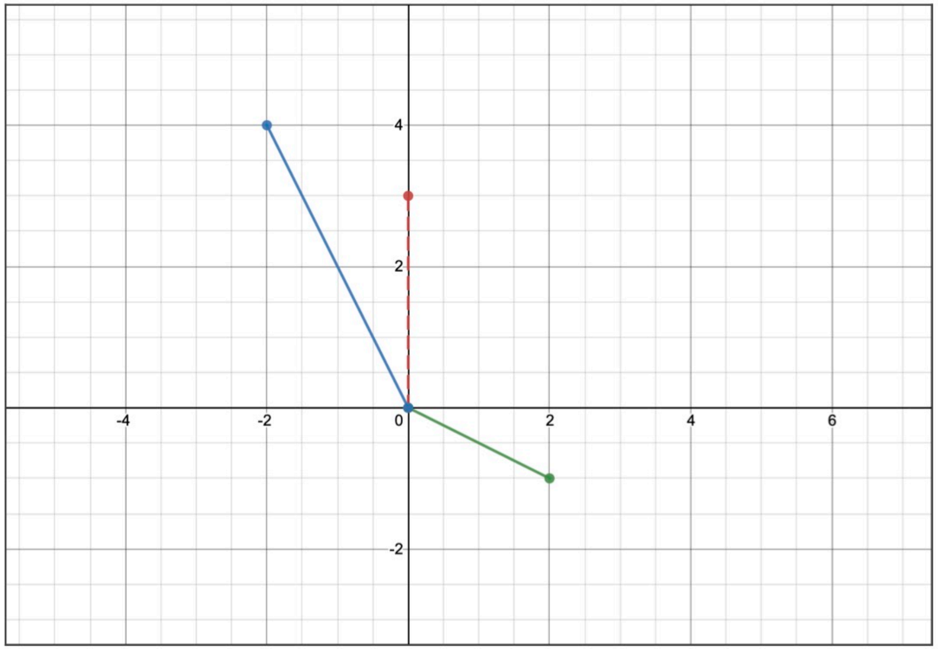
We begin by sketching out the respective straight lines $2x - y = 0$ and $-x + 2y = 3$. The solution of this system is the *intersection point* of these two straight lines, $x = 1,\ y = 2$. If we express this system of linear equations in matrix-vector notation, then we have $A\mathbf{x} = \mathbf{b}$, where $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ is the *coefficient matrix*, $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, and $\mathbf{b} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$. The system (i) corresponds to the *row picture,* and the solution strategy presented above lends a geometrical interpretation of this *row picture*.



There is an alternative (*and sometimes more useful*) geometrical picture, the *column picture,* which lends a different interpretation of the situation at hand. By careful inspection, we notice that the above system of equations can be re-written as follows:

$$x\begin{pmatrix} 2 \\ -1 \end{pmatrix} + y\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}.$$

Here the system of equations is expressed in terms of a *linear combination* of the column vectors of A. If we treat the columns as vectors in 2D Euclidean space, and consider the correct solutions (say we somehow know that $x = 1$ and $y = 2$), then we have the following picture. The resultant of adding the two vectors on the left hand side is identically equal to the vector $\mathbf{b}$ on the right hand side.

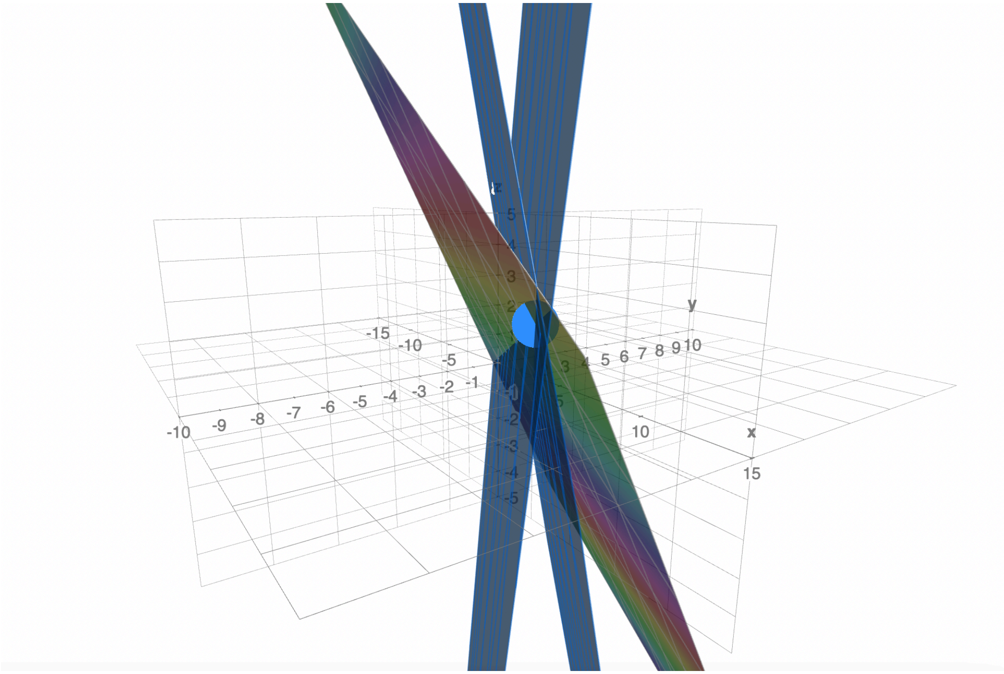


Throughout our study about linear algebra, the idea of *linear combination* and *column vectors of a matrix* will play a very important role in terms of the mathematical machinery as well as interpretation of the physical picture.

Of course, the question arises whether we can always solve this system as follows: $\mathbf{x} = A^{-1}\mathbf{b}$ for any given 2D vector $\mathbf{b}$? This will be possible only if A is invertible! In that case, the columns of A will span the entire 2D Euclidean plane (and $\mathbf{x} = A^{-1}\mathbf{b}$ will be the solution for any 2D vector $\mathbf{b}$). This should lead us to ask when is A invertible? i.e. When will the columns of A span[4] the entire 2D Euclidean plane? The answer should be obvious by inspecting the above 2D graph: whenever the columns of A are linearly independent.

eg. 4.2: Consider the system of linear equations:

$$\begin{aligned} 2x + 8y + 4z &= 2 \\ 2x + 5y + z &= 5 \\ 4x + 10y - z &= 1 \end{aligned} \qquad \text{..................................} \ \text{(ii)}$$

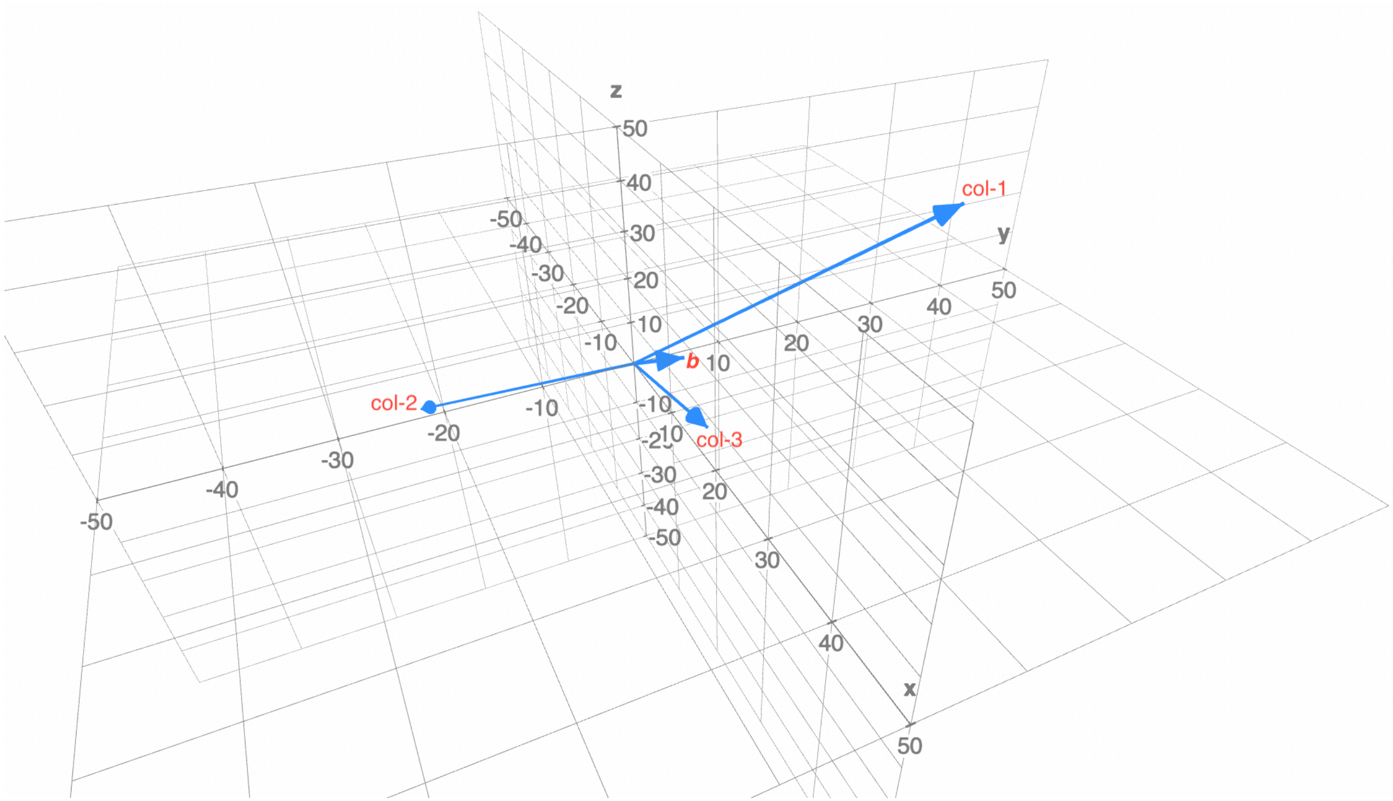


[4] *spanning a plane* here is analogous to obtaining any vector in the plane by a *linear combination of vectors.*

This system can be expressed in matrix form as: $A\mathbf{x} = \mathbf{b}$ where $A = \begin{pmatrix} 2 & 8 & 4 \\ 2 & 5 & 1 \\ 4 & 10 & -1 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, and $\mathbf{b} = \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}$. Here A is called the <u>*coefficient matrix*</u>. Three planes defined by the system of equations (ii) intersect at a point $x = 11,\ y = -4,\ z = 3$ which is the solution. This is the *row picture.*

Now let us examine the *column picture!*

$$x\begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix} + y\begin{bmatrix} 8 \\ 5 \\ 10 \end{bmatrix} + z\begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$$



Like in the previous example, a given 3D-vector $\mathbf{b}$ can be obtained by the linear combination of the column vectors of A with the appropriate coefficients (x, y, z). The appropriate coefficients on the l.h.s. form the solution set. This *column picture* and the *linear combination of the columns* allow us to determine, geometrically, the conditions when a unique solution can be attained for any 3D-vector $\mathbf{b}$. The answer is, once again, when A is invertible, i.e. when the columns of A are linearly independent (or equivalently when the column vectors of A are *not* co-planar[5]).

A peek into the future

Note that the route to obtain the solution involves finding the inverse of A which may be difficult especially if the number of unknown variables (and thereby the number of equations) are large. Computing the inverse of a large matrix becomes simpler by transforming the original coefficient matrix into a certain *reduced row-echelon form* which lies at the heart of several numerical techniques to solve systems of linear equations as well as characterises several important features of the coefficient matrix.

In any case, as a prelude to what is to come soon, the solution to this system of linear equations can be obtained by transforming the *augmented matrix* $\tilde{A}$, *where* $\tilde{A} = \begin{pmatrix} 2 & 8 & 4 & 2 \\ 2 & 5 & 1 & 5 \\ 4 & 10 & -1 & 1 \end{pmatrix}$, into its *reduced row-echelon form* $rref(\tilde{A}) = \begin{pmatrix} 1 & 0 & 0 & 11 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 3 \end{pmatrix}$. From the latter, the solutions can be directly obtained as $x = 11,\ y = -4,\ z = 3$. We will study in the next page about rref!

Why this turns out to be the case? We will study in a subsequent series of lectures! The point here is that the matrix structure and its rules (laws of matrix algebra) become very useful machinery in solving such systems of linear equations. We will later do a laboratory project to further understand the power of this technique to solve engineering problems.

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[5] *if the column vectors of A were co-planar, then a linear combination of them will not span the entire 3D plane and hence there would be some $\mathbf{b}$ that will be pathological!*