Tutorial Worksheet-5 (WL6.1)
Eigenvalues, Eigenvectors, Diagonalisation of matrices, LU Decomposition

Name and section: ______________________________

Instructor's name: ______________________________

1. Find eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 5 & -4 \\ 2 & -1 \end{bmatrix}$. Check whether the given matrix is diagonalizable or not? If it is diagonalizable, then express the given matrix in form $S^{-1}DS$.

Solution: The characteristic equation of the matrix $A = \begin{bmatrix} 5 & -4 \\ 2 & -1 \end{bmatrix}$ is given by

$$\det(A - \lambda I) = 0.$$

Now $\det(A - \lambda I) = 0$ implies

$\det \begin{bmatrix} 5-\lambda & -4 \\ 2 & -1-\lambda \end{bmatrix} = 0$
$\implies (5-\lambda)(-1-\lambda) + 8 = 0$
$\implies \lambda^2 - 4\lambda + 3 = 0$
$\implies (\lambda - 3)(\lambda - 1) = 0$
$\implies \lambda = 1, 3$

Therefore 1 and 3 are the eigenvalues of the matrix A each one of them having algebraic multiplicity 1.

Eigenvectors corresponding to the eigenvalue 1 are given by nonzero solutions of the following system of equations

$$(A - 1I)X = 0.$$

Now $(A - I)X = 0$ implies that $\begin{bmatrix} 4 & -4 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, i.e., $x - y = 0$ or $x = y$. Hence a solution of the above equation takes the form $\begin{bmatrix} y \\ y \end{bmatrix}$ or $y \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, where $y \in \mathbb{R}$. Hence any nonzero multiple of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of A corresponding to the eigenvalue 1. Therefore the geometric multiplicity of the eigenvalue 1 is 1. In particular, we choose $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as an eigenvector of A corresponding to the eigenvalue 1.

Since algebraic multiplicity and geometric multiplicity of every eigenvalue coincides we conclude that A is diagonalizable.

Eigenvectors corresponding to the eigenvalue 3 is given by nonzero solutions of the following system of equations

$$(A - 3I)X = 0.$$

Now $(A-3I)X = 0$ implies that $\begin{bmatrix} 2 & -4 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, i.e., $x - 2y = 0$ or $x = 2y$. Hence a solution of the above equation takes the form $\begin{bmatrix} 2y \\ y \end{bmatrix}$ or $y \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, where $y \in \mathbb{R}$. Hence any nonzero multiple of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector of A corresponding to the eigenvalue 3. Thus the geometric multiplicity of the eigenvalue 3 is 1. In particular, we choose $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ as an eigenvector of A corresponding to the eigenvalue 3.

Consider the 2×2 matrices P and D given by

$$P = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}.$$

Now note that $AP = \begin{bmatrix} 5 & -4 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 6 \\ 1 & 3 \end{bmatrix}$ and $PD = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 6 \\ 1 & 3 \end{bmatrix}$. Hence we obtain $AP = PD$. Since $\det(P) = -1$, P is an invertible matrix. Therefore we can also express $AP = PD$ as $A = PDP^{-1}$. Take $S = P^{-1} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$, so as to get the required form, i.e., $A = S^{-1}DS$.

2. Check whether the given matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ is diagonalizable or not?

Solution: Characteristic equation of A is given by

$$\det(A - \lambda I) = 0.$$

Now $\det(A - \lambda I) = 0$ implies

$\det \begin{bmatrix} 1-\lambda & 1 & 1 \\ 0 & -\lambda & 1 \\ 0 & 0 & 1-\lambda \end{bmatrix} = 0$

$\implies \lambda(1-\lambda)^2 = 0$

$\implies \lambda = 0, 1, 1$

This shows that 0 and 1 are the eigenvalues of A of algebraic multiplicity 1 and 2, respectively. Eigenvectors corresponding to the eigenvalue 1 is given by any nonzero solution of the following system of equations

$$(A - 1I)X = 0.$$

Now $(A-1I)X = 0$ implies that $\begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. RREF of the matrix $\begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ is clearly $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Solutions of the above system of equations is precisely given by the solutions of the system of equations $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Now note that rank of A is precisely the number of pivotal ones in its rref, which is 2, hence nullity$(A) = 3 - 2 = 1$. Thus the geometric multiplicity of 1 is 1. Since algebraic multiplicity of 1 is 2, it is not equal to the algebraic multiplicity of 1. Hence A is not diagonalizable.

3. Check whether the given matrix $A = \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix}$ is diagonalizable or not? If not, then justify your answer and if yes then find the matrix S such that $A = S^{-1}DS$

Solution: Characteristic equation of A is given by

$$\det(A - \lambda I) = 0.$$

Now $\det(A - \lambda I) = 0$ implies

$$\det \begin{bmatrix} -1-\lambda & -1 & -1 \\ -1 & -1-\lambda & -1 \\ -1 & -1 & -1-\lambda \end{bmatrix} = 0$$

$\implies (-1-\lambda)[(-1-\lambda)^2 - 1] + 1[(1+\lambda) - 1] - [1 - (1+\lambda)] = 0$

$\implies -(1+\lambda)[(1+\lambda)^2 - 1] + [(1+\lambda) - 1] - [1 - (1+\lambda)] = 0$

$\implies \lambda^2(\lambda + 3) = 0 \implies \lambda = 0, 0, -3$

This shows that 0 and -3 are the eigenvalues of A with algebraic multiplicity 2 and 1, respectively.

Eigenvectors corresponding to the eigenvalue 0 is given by nonzero solutions of the following system of equations

$$(A - 0I)X = 0.$$

Now $AX = 0$ implies that $\begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, i.e., $x+y+z = 0$ or $z = -x-y$. Hence a solution of the above equation takes the form $\begin{bmatrix} x \\ y \\ -x-y \end{bmatrix}$ or $x \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, where $x, y \in \mathbb{R}$.

Hence any nonzero element in the two dimensional space spanned by $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$ is an eigenvector of A corresponding to the eigenvalue 0. In particular, we choose $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ so as to find linearly independent eigenvectors of A corresponding to the eigenvalue 0. Thus the geometric multiplicity of the eigenvalue 0 is 2.

Eigenvectors corresponding to the eigenvalue -3 is given by nonzero solutions of the following system of equations

$$(A+3I)X = 0.$$

Now $(A+3I)X = 0$ implies that $\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. RREF of the matrix $\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$ is clearly $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$. Solution of the above system of equations is precisely given by the solutions of the system of equations $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Solving we obtain $x = z$ and $y = z$, so as to find the general solution of the original equation of the form $z \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, where $z \in \mathbb{R}$. Hence the geometric multiplicity of -3 is 1. In particular, we choose $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ as an eigenvector of A corresponding to the eigenvalue -3.

Let $P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix}$. Then $AP = \begin{bmatrix} 0 & 0 & -3 \\ 0 & 0 & -3 \\ 0 & 0 & -3 \end{bmatrix}$ and $PD = \begin{bmatrix} 0 & 0 & -3 \\ 0 & 0 & -3 \\ 0 & 0 & -3 \end{bmatrix}$. Since $\det(P) = 2 + 1 = 3$, P is an invertible matrix. Therefore $AP = PD$ can be expressed as $P^{-1}AP = D$ and this shows that A is diagonalizable. Take $S = P^{-1} = \frac{1}{3} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ 1 & 1 & 1 \end{bmatrix}$ to get the desired form in the question.

4. Find the eigen basis and its dimension of the matrix $\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 2 & 2 & 0 \end{bmatrix}$

Solution: Characteristic equation of A is given by

$$\det(A - \lambda I) = 0.$$

Now $\det(A - \lambda I) = 0$ implies

$\det \begin{bmatrix} 1-\lambda & 1 & 0 \\ 0 & -1-\lambda & -1 \\ 2 & 2 & -\lambda \end{bmatrix} = 0$

$\implies (1-\lambda)[(-1-\lambda)(-\lambda) + 2] - 2$

$\implies (1-\lambda)[2 + \lambda + \lambda^2] - 2$

$\implies \lambda(\lambda^2 + 1) = 0 \implies \lambda = 0, i, -i.$

This shows that $0, i$ and $-i$ are the eigenvalues of A each one of them having algebraic multiplicity 1.

Eigenvectors corresponding to the eigenvalue 0 is given by nonzero solutions of the following system of equations

$$(A - 0I)X = 0.$$

Now $AX = 0$ implies that $\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. The RREF of $\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 2 & 2 & 0 \end{bmatrix}$ is given by $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Solution of the above system of equations is precisely given by the solutions of the system of equations $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Solving we obtain $x = z$ and $y = -z$, so as to find the general solution of the original equation of the form $z \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, where $z \in \mathbb{R}$. Hence the geometric multiplicity of the eigenvalue 0 is 1.

Eigenvectors corresponding to the eigenvalue i is given by nonzero solutions of the following system of equations

$$(A + iI)X = 0.$$

Now $(A + iI)X = 0$ implies that $\begin{bmatrix} 1-i & 1 & 0 \\ 0 & -1-i & -1 \\ 2 & 2 & -i \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. RREF of the matrix $\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$ is clearly $\begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{1-i}{2} \\ 0 & 0 & 0 \end{bmatrix}$. Solution of the above system of equations is precisely given by the solutions of the system of equations $\begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{1-i}{2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Solving we obtain $x = \frac{1}{2}z$ and $y = \frac{i-1}{2}z$, so as to find the general solution of the original equation of the form $z \begin{bmatrix} 1 \\ -1+i \\ 2 \end{bmatrix}$, where $z \in \mathbb{C}$. In particular we choose $\begin{bmatrix} 1 \\ -1+i \\ 2 \end{bmatrix}$ as an eigenvector of A corresponding to the eigenvalue i.

In order to find an eigenvector of A corresponding to the eigenvalue $-i$ we use the previous computation in the following way. We already found $X = \begin{bmatrix} 1 \\ -1+i \\ 2 \end{bmatrix} \in \mathbb{C}^3$ such that $AX = iX$. Taking conjugate on both side we obtain $\overline{AX} = \overline{iX}$. Since A is a real matrix, $A\bar{X} = -i\bar{X}$. Hence $\bar{X} = \begin{bmatrix} 1 \\ -1-i \\ 2 \end{bmatrix}$ is an eigenvector of A corresponding to the eigenvalue $-i$.

Thus we conclude that all eigenvalues are of algebraic multiplicity and geometric multiplicity 1 over the field of complex numbers.

5. Check whether

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}.$$

is diagonalizable or not. If it is diagonalizable then using the diagonal matrix find the rank and determinant of A. Also find A^5.

Solution: Characteristic equation of A is given by

$$\det(A - \lambda I) = 0.$$

Now $\det(A - \lambda I) = 0$ implies

$$\det \begin{bmatrix} 1-\lambda & 2 & 3 \\ 2 & 4-\lambda & 6 \\ 3 & 6 & 9-\lambda \end{bmatrix} = 0$$

$\implies (1-\lambda)[(4-\lambda)(9-\lambda) - 36] - 2[2(9-\lambda) - 18] + 3[12 - 3(4-\lambda)] = 0$

$\implies (1-\lambda)[\lambda^2 - 13\lambda] + 13\lambda = 0$

$\implies \lambda^2 - 13\lambda - \lambda^3 + 13\lambda^2 + 13\lambda = 0$

$\implies \lambda^3 - 14\lambda^2 = 0$

$\implies \lambda = 0, 0, 14$

This shows that 0 and 14 are the eigenvalues of A with algebraic multiplicity 2 and 1, respectively.

Eigenvectors corresponding to the eigenvalue 0 is given by nonzero solutions of the following system of equations

$$(A - 0I)X = 0.$$

Now $AX = 0$ implies that $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. RREF of $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$ is given by $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Solution of the above system of equations is precisely given by the solutions of the system of equations $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Solving we obtain $x = -2y - 3z$, so as to find the general solution of the original equation of the form $y \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$, where $y, z \in \mathbb{R}$. Hence the geometric multiplicity of the eigenvalue 0 is 2. In particular, we choose $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$ as linearly independent eigenvectors of A corresponding to the eigenvalue 0.

Eigenvectors corresponding to the eigenvalue 14 is given by nonzero solutions of the following system of equations

$$(A - 14I)X = 0.$$

Now $(A - 14I)X = 0$ implies that $\begin{bmatrix} -13 & 2 & 3 \\ 2 & -10 & 6 \\ 3 & 6 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. RREF of $\begin{bmatrix} -13 & 2 & 3 \\ 2 & -10 & 6 \\ 3 & 6 & -5 \end{bmatrix}$ is given by $\begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 0 \end{bmatrix}$. Solutions of the above system of equations is precisely given by the solutions of the system of equations $\begin{bmatrix}1 & 0 & -\frac{1}{3}\\ 0 & 1 & -\frac{2}{3}\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$. Solving we obtain $x = \frac{1}{3}z, y = \frac{2}{3}z$, so as to find the general solution of the original equation of the form $z\begin{bmatrix}1\\ 2\\ 3\end{bmatrix}$, where $z \in \mathbb{R}$. Hence the geometric multiplicity of the eigenvalue 14 is 1. In particular, we choose $\begin{bmatrix}1\\ 2\\ 3\end{bmatrix}$ to be an eigenvectors of A corresponding to the eigenvalue 14.

Let $P = \begin{bmatrix}-2 & -3 & 1\\ 1 & 0 & 2\\ 0 & 1 & 3\end{bmatrix}$ and $D = \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 14\end{bmatrix}$. We obtain $AP = \begin{bmatrix}0 & 0 & 14\\ 0 & 0 & 28\\ 0 & 0 & 42\end{bmatrix}$ and $PD == \begin{bmatrix}0 & 0 & 14\\ 0 & 0 & 28\\ 0 & 0 & 42\end{bmatrix}$. Since $\det(P) = 14$, P is invertible and hence we obtain $P^{-1}AP = D$. Therefore $rank(A) = rank(D) = 1$. Moreover $\det(A) = \det(D) = 0$. Also since $P^{-1}AP = D$, we have $P^{-1}A^5P = D^5$, i.e., $A^5 = PD^5P^{-1}$. We know $P = \begin{bmatrix}-2 & -3 & 1\\ 1 & 0 & 2\\ 0 & 1 & 3\end{bmatrix}$, $D^5 = \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 537824\end{bmatrix}$. Since $P^{-1} = \frac{1}{14}\begin{bmatrix}-2 & 10 & -6\\ -3 & -6 & 5\\ 1 & 2 & 3\end{bmatrix}$, $A^5 = 14^4A$.

6. Find the solution of system of equations

$$\begin{aligned} x - 3y + 5z &= 1\\ 2x - 4y + 7z &= 1\\ -x - 2y + z &= 1 \end{aligned}$$

using LU factorization method.

Solution: Given system of equations can be written in matrix form as

$$AX = B \tag{0.1}$$

where

$$A = \begin{bmatrix}1 & -3 & 5\\ 2 & -4 & 7\\ -1 & -2 & 1\end{bmatrix},\ X = \begin{bmatrix}x\\ y\\ z\end{bmatrix},\ B = \begin{bmatrix}1\\ 1\\ 1\end{bmatrix}$$

Now, the matrix A can be factorized to LU form as per below strategy:
Firstly, reduce the matrix A to triangular form by the elementary row operations as follows:

$$\begin{bmatrix}1 & -3 & 5\\ 2 & -4 & 7\\ -1 & -2 & 1\end{bmatrix} \underset{R_2\to -2R_1+R_2}{\xrightarrow{R_3\to R_1+R_3}} \begin{bmatrix}1 & -3 & 5\\ 0 & 2 & -3\\ 0 & -5 & 6\end{bmatrix} \xrightarrow{R_3\to\frac{5}{2}R_2+R_3} \begin{bmatrix}1 & -3 & 5\\ 0 & 2 & -3\\ 0 & 0 & -\frac{3}{2}\end{bmatrix}$$

Hence, these operations yield the matrix $U = \begin{bmatrix} 1 & -3 & 5 \\ 0 & 2 & -3 \\ 0 & 0 & -\frac{3}{2} \end{bmatrix}$ and $L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -\frac{5}{2} & 1 \end{bmatrix}$
The entries 2,-1,$-\frac{5}{2}$ in L are the negatives of the multipliers $-2, 1, \frac{5}{2}$ in the above row operations. Inserting the factorization of $A = LU$ into an expression (0.1), we get

$$LUX = B \tag{0.2}$$

with

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -\frac{5}{2} & 1 \end{bmatrix}, \ U = \begin{bmatrix} 1 & -3 & 5 \\ 0 & 2 & -3 \\ 0 & 0 & -\frac{3}{2} \end{bmatrix}$$

Assume $UX = Y$, where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$, Hence the expression (0.2) becomes

$$LY = B \tag{0.3}$$

Re-writting the equation (0.3) into the system form as follows:

$$\begin{aligned} y_1 &= 1 \\ 2y_1 + y_2 &= 1 \\ -y_1 - \frac{5}{2}y_2 + y_3 &= 1 \end{aligned}$$

Using forward substitution, the values of y_1, y_2, y_3 are

$$y_1 = 1, \ y_2 = -1, \ y_3 = -\frac{1}{2}$$

Inserting the values of y_1, y_2, y_3 into $UX = Y$, and writting them into system form, we have:

$$\begin{aligned} x - 3y + 5z &= 1 \\ 2y - 3z &= -1 \\ -\frac{3}{2}z &= -\frac{1}{2} \end{aligned}$$

Solving, the above system using backward substitution, the values of x, y, z are

$$x = -\frac{2}{3}, \ y = 0, \ z = \frac{1}{3}$$