Recap of Last Lecture

finite (eg. 4-digit arithmetic)

→ Using the same computer (w/ fixed precision), different numerical strategies (algos.) incur different error while performing the same numerical task.

→ further do the following reading assignment p-25, example 6 from the textbook (Burden & Faires)

↳ This example shows "nested" strategy to evaluate f's incur less error!

→ Forward error vs Backward error

Stability & convergence of numerical algorithms — a general overview

* Numerical Algos. will invariably incur error

* $e_0 > 0$ (in magnitude) — initial error (eg. $|y - y^*|$)

e_n — error after performing n iterations of the algo.

STABLE algorithm

Acceptable

If $e_n \approx C n e_0$; where C is const. not dependent on n

linear growth of error if C, e_0 are small

unSTABLE algorithm

to be avoided

If $e_n \approx C^n e_0$; $C > 1$ then exponential growth of error!

* Small changes in i/p data causes small changes in results (estimates); then algorithm is **STABLE**.

else **unSTABLE**!

* **Conditionally stable** :- Certain algos. are stable for some data set & unstable for other types of data set.

Condition number: Ratio of fwd. error to bkwd error for small changes in the problem statement.

eg. Consider the root finding problem $f(x) = 0$ from last class.

$$\text{Cond}^n \text{ no.} = \frac{\text{fwd. error}}{\text{bkwd error}} = \frac{|x^* - x_0|}{|f(x^*) - f(x_0)|} = \frac{(x_0 + \Delta) - x_0}{f(x_0 + \Delta) - f(x_0)}$$

Contd...

$$\text{Cond}^{n}\ \text{no.}\ (K(x_0)) = \frac{\Delta}{\Delta f'(x_0)} = \frac{1}{f'(x_0)}$$

b/c Taylor expanding $f(x)$ about x_0:

$$f(x_0+\Delta) = f(x_0) + \Delta f'(x_0) + \text{H.O.T.}$$

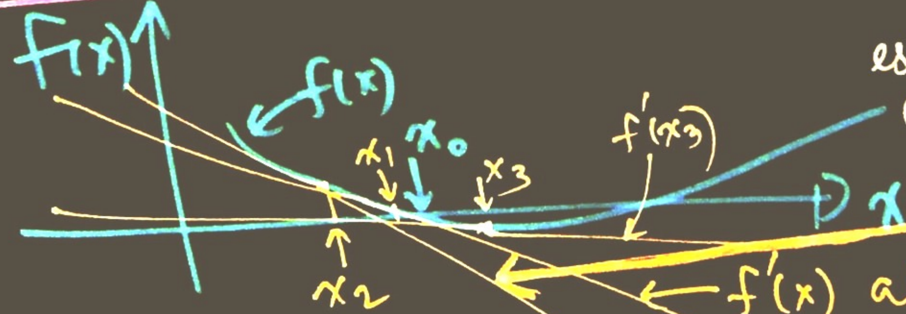
$$\Rightarrow f(x_0+\Delta) - f(x_0) \cong \Delta f'(x_0)$$

Now since we donot know x_0; we cannot calculate $f'(x_0)$!

$$\boxed{\frac{1}{M_1} > K = \frac{1}{f'(x_0)} > \frac{1}{M_2}}$$

But we know $f(x)$ in general & may be able to bound $f'(x)$ near $x = x_0$.

$\left\{ \begin{array}{l} f'(x_0) > M_1 \\ f'(x_0) < M_2 \end{array} \right.$ by estimate analyzing slopes of tangent lines near x_0 of



$f(x)$ at $x = x_2$

$f'(x)$ at $x = x_1$ (Close to x_0)

Convergence

(4)

Let $\{\beta_n\}_{n=1}^{\infty} \longrightarrow 0$ like $\frac{1}{n^p} \to 0$.

and $\{\alpha_n\}_{n\geq 1} \longrightarrow \alpha$.

If $\exists$ $K \neq 0$ s.t.

$$|\alpha_n - \alpha| \leq K|\beta_n| \quad \text{for large } n;$$

then $\{\alpha_n\}_{n\geq 1} \longrightarrow \alpha$ w/ rate of convergence $O(\beta_n)$ or $O\left(\frac{1}{n^p}\right)$

* we are generally interested in the largest such value of p.

example of convergence.

Q) Let $\alpha_n = \frac{n+1}{n^2}$; $\gamma_n = \frac{n+3}{n^3}$; $n \geq 1$

Clearly $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \gamma_n = 0$!

$$\lim_{n\to\infty} \frac{n+1}{n^2} = \lim_{n\to\infty} \frac{\frac{n}{n^2}+\frac{1}{n^2}}{(n^2/n^2)} = \lim_{n\to\infty} \frac{\frac{1}{n}+\frac{1}{n^2}}{1} = 0$$

Find / Compare the rate of convergence of α_n and γ_n to 0?

Ans:- Let $\beta_n = \frac{1}{n}$ and $\tilde{\beta}_n = \frac{1}{n^2}$.

$$|\alpha_n - 0| = \frac{n+1}{n^2} \leq \frac{n+n}{n^2} = 2\left(\frac{1}{n}\right) = 2\beta_n \text{ i.e. } |\alpha_n - 0| \leq 2|\beta_n|$$

and

$$|\gamma_n - 0| = \frac{n+3}{n^3} \leq \frac{n+3n}{n^3} = 4\frac{1}{n^2} = 4\tilde{\beta}_n \text{ i.e. } |\gamma_n - 0| \leq 4|\tilde{\beta}_n|$$

$\therefore \{\alpha_n\} \to 0$ w/ $O\left(\frac{1}{n}\right)$ and $\{\gamma_n\} \to 0$ w/ $O\left(\frac{1}{n^2}\right)$.