Lec 17: Numerical computation of EVs and eVs DATE 01/04/2025

Rayleigh Quotient

Q) Why does the iterative Power method to find EVs converge to the dominant EV?

Ans:-

Let $A \in M_{n \times n}(\mathbb{F})$ w/ a complete set of n linearly independent EVs $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ w/ corresponding eVs $\lambda_1, \lambda_2, \ldots, \lambda_n$

then any initial vector $\vec{x}_0$ in $\mathbb{R}^n$ can be expressed as a linear combination

$$\bar{x}_0 = c_1\bar{v}_1 + c_2\bar{v}_2 + \cdots + c_n\vec{v}_n$$

$$A\bar{x}_0 = c_1 A\bar{v}_1 + c_2 A\bar{v}_2 + \cdots + c_n A\vec{v}_n$$

$$= c_1\lambda_1\bar{v}_1 + c_2\lambda_2\bar{v}_2 + \cdots + c_n\lambda_n\vec{v}_n$$

$$A^2\vec{x}_0 = c_1 A(\lambda_1\vec{v}_1) + c_2 A(\lambda_2\vec{v}_2) + \cdots + c_n A(\lambda_n\vec{v}_n)$$

$$= c_1\lambda_1^2\bar{v}_1 + \cdots + c_n\lambda_n^2\vec{v}_n$$

finally, $A^K\vec{x}_0 = c_1\lambda_1^K\vec{v}_1 + c_2\lambda_2^K\bar{v}_2 + \cdots + c_n\lambda_n^K\vec{v}_n$

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Since, by convention, we have assumed the dominant EV as λ_1

i.e. $|\lambda_1| > |\lambda_i| \; \forall i \neq 1$

$$A^K \vec{x}_0 = \lambda_1^K \left(c_1 \vec{v}_1 + c_2 \left(\frac{\lambda_2}{\lambda_1}\right)^K \vec{v}_2 + \dots + c_n \left(\frac{\lambda_n}{\lambda_1}\right)^K \vec{v}_n \right)$$

as $K \to \infty$ $\left(\frac{\lambda_i}{\lambda_1}\right)^K \to 0 \; \forall i \neq 1$

b/c $\left|\frac{\lambda_i}{\lambda_1}\right| < 1 \; \forall i \neq 1$

$$\therefore A^K \vec{x}_0 \approx \underbrace{(\lambda_1^K c_1)}_{\text{Scaling factor}} \vec{v}_1$$

dominant EV

If A has complex EVs; we may require additional cond^n^s!

$\therefore$ the Power method surely converges to the dominant EV. (esp. when A is a diagonalisable matrix) provided

(I) A has one strict dominant ev #

(II) Initial vector $\vec{x}_0$ has a component in the dir^n^ of $\vec{v}_1$ (i.e. $c_1 \neq 0$)