

# Application of Harmonic Series

eg 1) How often are weather records broken?

Suppose we have rainfall data for 100 yrs. How many record breaking falls of rain do you expect have taken place over that period?

Assume: - rainfall (amt) is random  
i.e. amt of rain on a given yr is independent of (has no influence on) the rainfall amt in any subsequent yr.

<del>Yr</del>	record breaking yr for sure or w/ prob.	Expected no. of record yrs in 1st n yrs
$n = 1$	Yes	1
$n = 2$	Yes w/ probability $\frac{1}{2}$ b/c of assumption	$1 + \frac{1}{2}(1) + \frac{1}{2}(0)$ (1) is for Yes (0) is for No.
$n = 3$	Yes w/ probability $\frac{1}{3}$ b/c of assumption	$1 + \frac{1}{2} + \frac{1}{3}$
$n = k$	Yes w/ probability $\frac{1}{k}$	$1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{k}$

So for 100 yrs the sum  $(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{100}) \approx 5.19$

the no. of record years in an do of observations is 6  
→ Corroborates our intuition about the divergence of Harmonic Series.

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So above, we have seen an example where the Harmonic Series shows up.

But if the harmonic series is divergent, then is it of any use? After all, what's the use for anything that blows up? (divergent).

Let's try to address this question with the following example.

eg(2)

Crossing the desert during the Gulf war posed many problems - one of which is described here.

You have to cross the desert in a jeep. There are no sources of fuel in the desert (no gas pumps/petrol stations) & you cannot carry enough fuel in the jeep to cross the desert in one go. You do not have time to establish enough fuel dumps but you do have a very 'large' supply of jeeps.

Q) How can you get across the desert using the minimum amount of fuel?

19②

Let us define the maximum distance one jeep can travel on full tank as one a distance of one tankful.

If 2 jeeps set out together, each travels  $\frac{1}{3}$  tankful unit of distance; then jeep 2 transfers  $\frac{1}{3}$  of <sup>its</sup> tank to jeep 1 & returns to base on the remaining  $\frac{1}{3}$  tank.

Jeep 1 is then able to travel a total of  $\frac{1}{3} + \frac{1}{3}$  tankful units of distance.

from its own tank      borrowed fuel from jeep 2.

w/ three jeeps; stop after  $\frac{1}{5}$  tankful units of distance & transfer  $\frac{1}{5}$  of a tankful to each of jeeps 2 & 3, which are now full again. Jeep 3 now has  $\frac{2}{5}$  tank of fuel. Jeep 1 & 2 proceed as before like the case of  $n=2$  jeeps & jeep 2 returns to jeep 3 empty tank but bet'n them they have enough fuel to return to base. Jeep 1, meanwhile, has travelled a total of  $1 + \frac{1}{3} + \frac{1}{5}$  tankful of distance.

from its own tank      from jeep 2      from jeep 3

w/  $n=4$  jeeps; jeep 1 can get us to a 19③

total distance of  $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7}$  tankful distance

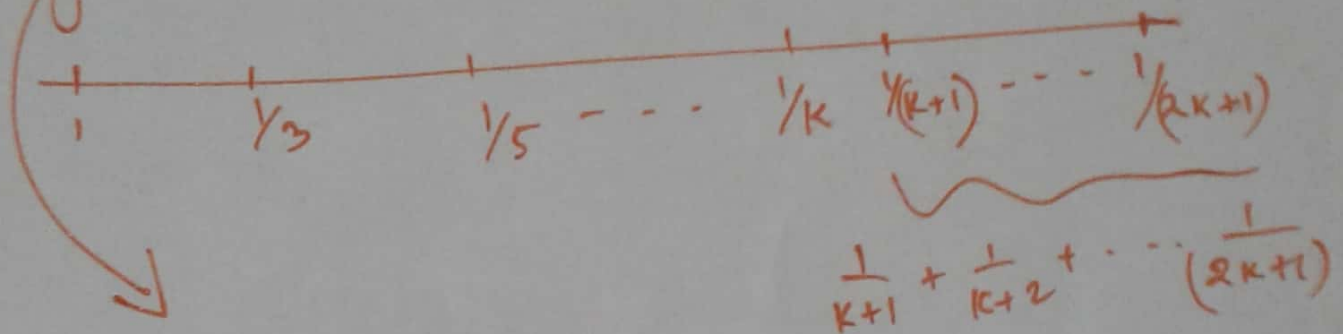
$n = k$  jeeps get us as far as

$$1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots + \frac{1}{(2k+1)} \text{ tankful distance}$$

Clearly  $1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{(2k+1)} > \frac{1}{2} (1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{k} + \frac{1}{k+1})$

or  $2 [1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{(2k+1)}] > (1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{k} + \frac{1}{k+1})$

Why?



$$1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{(2k+1)} > \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \dots + \frac{1}{2k}$$

& term by term

$$1 > \frac{1}{2}$$

$$\frac{1}{3} > \frac{1}{4}$$

$$\frac{1}{5} > \frac{1}{6}$$

⋮

$$\frac{1}{(2k+1)} > \frac{1}{2k+2}$$

&  $\sum_{m=1}^{\infty} \frac{1}{m}$  diverges  $\Rightarrow \sum_{k=0}^{\infty} \frac{1}{(2k+1)}$  diverges

So w/ an  $\infty$  supply of jeeps we can

(very large)  
traverse an arbitrarily large  $\infty$   
desert! - by using the system/technology  
of fuel transfers.

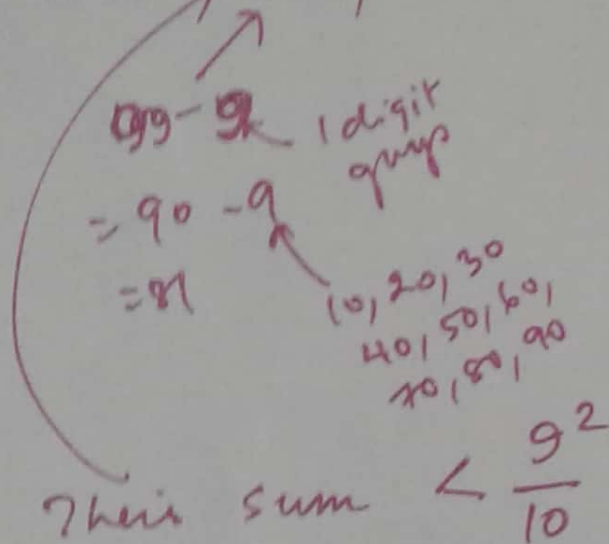
Q3) # On similar lines, it can be  
shown the series of reciprocals of prime  
is also divergent

i.e.  $\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \dots \rightarrow$  div.

Q4) Instead of deleting "non-prime"  
denominators from the harmonic  
series; we delete every term  
which has a zero in its denominator  
thus it seems a reasonable guess  
that 1 in every 10 terms is deleted  
from the harmonic series - - - so a good  
guess would be that the new series  
must also diverge but our guess  
turns out to be wrong!

Let us look first at all terms of the leftover series w/ just 1 digit in the denominator there are exactly 9 of these terms & they are all less than 1  
 eg to  $1 \leq 1, \frac{1}{2} \leq 1, \dots$   
 their sum is  $< 9$   $\frac{1}{9} \leq 1$

Next, look at terms of the leftover series w/ exactly 2 digits in denominator there are 81 of these, all  $< \frac{1}{10}$



In general; there are  $9^k$  terms of the series w/ exactly  $k$  digits in the denominator (after removing the 0 digit terms from  $D^r$ )

& each is less than  $\frac{1}{10^{k-1}}$

Their total is less than  $\frac{9^k}{10^{k-1}}$ .

$\therefore$  Sum of the terms of the leftover series is less than  $9(1 + \frac{9}{10} + (\frac{9}{10})^2 + \dots)$

Which is a geometric series whose  
sum to  $\infty$  terms is  $9 \frac{1}{1-r} = \frac{1}{1-9/10} \times 9$   
 $= 10 \times 9 = 90$

Thus the harmonic series w/o the terms  
containing 0 digits converges !!

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Check out Dresene's proof of div. of  
harmonic series.

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