Application of Harmonic Series

eg 1) How often are weather records broken?

Suppose we have rainfall data for 100 yrs. How many record breaking falls of rain do you expect have taken place over that period?

Assume :- rainfall (amt) is random
i.e amt of rain on a given yr is independent of (has no influence on) the rainfall amt in any subsequent yr.

yr	record breaking yr for sure or w/ prob	Expected no of record yrs in 1st n yrs
$n = 1$	Yes	1
$n = 2$	Yes w/ probability $1/2$ b/c of assumption	$1 + \frac{1}{2}(1) + \frac{1}{2}(0)$ (1) is for Yes (0) is for No.
$n = 3$	Yes w/ probability $1/3$ b/c yr 3 yr 1 yr 3 yr 2 yr 3	$1 + 1/2 + 1/3$
$n = k$	Yes w/ probability $1/k$	$1 + 1/2 + 1/3 + \cdots + 1/k$

So for 100 yrs the sum $\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{100}\right) \approx 5.19$ Ans 1

The no. of record years in an do of observation is 60
=> Corroborates our intuition about the divergence of Harmonic series.

So above, we have seen an example where the Harmonic series shows up.
But if the harmonic series is divergent, then is it of any use? After all, What's the use for anything that blows up? (divergent).

Let's try to address this question with the following example.

eg(2)

Crossing the desert during the Gulf war posed many problems - one of which is described here!

You have to cross the desert in a jeep. There are no sources of fuel in the desert (no gas pumps/petrol stations) & you cannot carry enough fuel in the jeep to cross the desert in one go. You do not have time to establish enough fuel dumps but you do have a very "large" supply of jeeps.

Q) How can you get across the desert using the minimum amount of fuel?

Let us define the maximum distance one jeep can travel on full tank as ~~one~~ a distance of one tankful.

If 2 jeeps set out together, each travels 1/3 tankful unit of distance; then jeep 2 transfers 1/3 of its tank to jeep 1 & returns to base on the remaining 1/3 tank.
Jeep 1 is then able to travel a total of $\underset{\text{from its own tank}}{1} + \underset{\text{borrowed fuel from jeep 2.}}{1/3}$ tankful units of distance.

W/ three jeeps; stop after 1/5 tankful units of distance & transfer 1/5 of a tankful to each of jeeps 2 & 1, which are now full again. Jeep 3 now has 2/5 tank of fuel. Jeep 1 & 2 proceed as before like the case of n = 2 jeeps & jeep 2 returns to jeep 3 empty tank but bet'n them they have enough fuel to return to base. Jeep 1, meanwhile, has travelled a total of $\underset{\text{from its own tank}}{1} + \underset{\text{from jeep 2}}{1/3} + \underset{\text{from jeep 3}}{1/5}$ tankful of distance.

W/ n = 4 jeeps; jeep 1 can get us to a

total distance of $1 + 1/3 + 1/5 + 1/7$ tankful distance

$n = k$ jeeps get us as far as

$1 + 1/3 + 1/5 + 1/7 + \cdots + 1/(2k+1)$ tankful distance.

Clearly $1 + 1/3 + 1/5 + \cdots + 1/(2k+1) > \frac{1}{2}\left(1 + 1/2 + 1/3 + \cdots + 1/k + 1/(k+1)\right)$

or $2\left[1 + 1/3 + 1/5 + \cdots + 1/(2k+1)\right] > \left(1 + 1/2 + 1/3 + \cdots + 1/k + 1/(k+1)\right)$

Why ?

$$\frac{1}{k+1} + \frac{1}{k+2} + \cdots \frac{1}{(2k+1)}$$

$$1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{(2k+1)} > \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{2k}$$

& term by term

$$1 > 1/2$$

$$1/3 > 1/4$$

$$1/5 > 1/6$$

$$\vdots$$

$$\frac{1}{(2k+1)} > \frac{1}{2k+2}$$

& $\sum_{m=1}^{\infty} \frac{1}{m}$ diverges $\Rightarrow \sum_{k=0}^{\infty} \frac{1}{(2k+1)}$ diverges

So w/ an ∞ supply of jeeps we can
(very large)
traverse an arbitrarily large (∞)
desert! – by using the system/technology
of fuel transfers.

Q3) # On similar lines, it can be
shown the series of reciprocals of prime
is also divergent

i.e. $\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \cdots \nearrow$ div.

Q4) Instead of deleting "non-prime"
denominators from the harmonic
series ; we delete every term
which has a zero in its denominator
thus it seems a reasonable guess
that 1 in every 10 terms is deleted
from the harmonic series ... so a good
guess would be that the new series
must also diverge but our guess
turns out to be wrong!

Let us look first at all terms of the leftover series w/ just 1 digit in the denominator there are exactly 9 of these terms & they are all less than eq to 1 b/c $1 \le 1$, $1/2 \le 1$, ... $1/9 \le 1$

Their sum is < 9

Next, look at terms of the left over series w/ exactly 2 digits in denominator there are 81 of these, all $< 1/10$

$99 - 9$ 1 digit grp

$= 90 - 9$

$= 81$

10, 20, 30, 40, 50, 60, 70, 80, 90

Their sum $< \frac{9^2}{10}$

In general; there are 9^k terms of the series w/ exactly k digits in the denominator (after removing the 0 digit terms from D')

& each is less than $1/10^{k-1}$

Their total is less than $9^k/10^{k-1}$.

$\therefore$ Sum of the terms of the leftover series is less than $9(1 + 9/10 + (9/10)^2 + \cdots)$

Which is a geometric series whose sum to ∞ terms is $9 \cdot \frac{1}{1-r} = \frac{1}{1-9/10} \times 9$

$= 10 \times 9 = 90$

Thus the harmonic series w/o the terms containing 0 digits converges !!

Check out Oresme's proof of div. of harmonic series.

#.