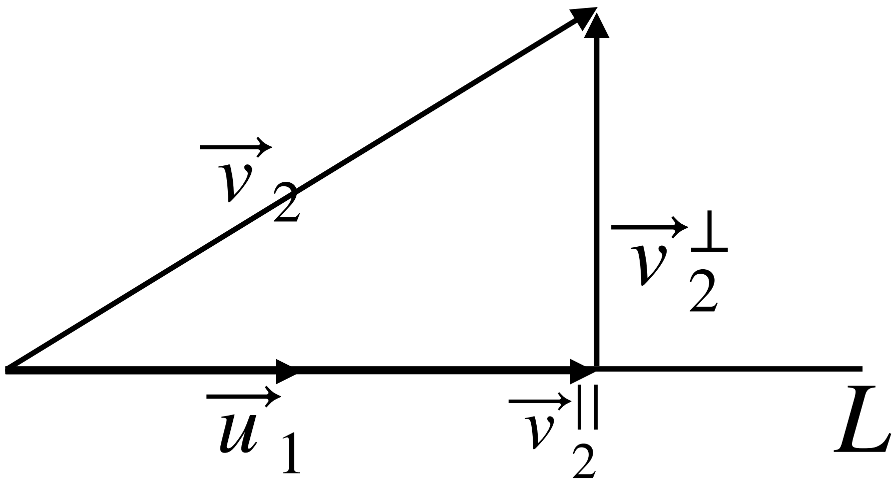
How to construct orthonormal basis vectors?

Let us illustrate this procedure by an example of a vector space V which is a plane. Given that V has a basis $\overrightarrow{v}_1$ and $\overrightarrow{v}_2$ which are **NOT** orthogonal, how do we construct an orthonormal basis?

Step 1: The first basis vector is easy to construct: $\overrightarrow{u}_1 = \dfrac{1}{||\overrightarrow{v}_1||}\overrightarrow{v}_1$.



Step 2: Construct $\overrightarrow{v}_2^{||} = \text{proj}_L(\overrightarrow{v}_2) = \langle \overrightarrow{u}_1, \overrightarrow{v}_2 \rangle \overrightarrow{u}_1$

Step 3: Construct $\overrightarrow{v}_2^{\perp} = \overrightarrow{v}_2 - \overrightarrow{v}_2^{||}$.

Step 4: Construct $\overrightarrow{u}_2 = \dfrac{1}{||\overrightarrow{v}_2^{\perp}||}\overrightarrow{v}_2^{\perp}$

Example: Find an orthonormal basis $\vec{u}_1, \vec{u}_2$ of the subspace $V = \text{span}\left(\begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\9\\9\\1\end{pmatrix}\right)$ of $\mathbb{R}^4$ with basis

$$\vec{v}_1 = \begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix}1\\9\\9\\1\end{pmatrix}.$$

Solution: Using the procedure mentioned above, we arrive at $\vec{u}_1 = \begin{pmatrix}1/2\\1/2\\1/2\\1/2\end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix}-1/2\\1/2\\1/2\\-1/2\end{pmatrix}$

Gram-Schmidt orthonormalisation process

Consider the basis $\vec{v}_1, \ldots, \vec{v}_m$ of a subspace V of $\mathbb{R}^n$. For $j = 2,\ldots,m$; we resolve the vector $\vec{v}_j$ into its components parallel and perpendicular to the span of the preceding vectors $\vec{v}_1, \ldots, \vec{v}_{j-1}$:

$$\vec{v}_j = \vec{v}_j^{||} + \vec{v}_j^{\perp}, \qquad \text{with respect to span}(\vec{v}_1, \ldots, \vec{v}_{j-1}).$$

Then $\vec{u}_1 = \dfrac{1}{||\vec{v}_1||}\vec{v}_1, \quad \vec{u}_2 = \dfrac{1}{||\vec{v}_2^{\perp}||}\vec{v}_2^{\perp}, \ldots, \vec{u}_j = \dfrac{1}{||\vec{v}_j^{\perp}||}\vec{v}_j^{\perp}, \ldots, \vec{u}_m = \dfrac{1}{||\vec{v}_m^{\perp}||}\vec{v}_m^{\perp}$

Is an orthonormal basis of V. Here $\vec{v}_j^{\perp} = \vec{v}_j - \vec{v}_j^{||} = \vec{v}_j - \langle \vec{u}_1, \vec{v}_j \rangle \vec{u}_1 - \cdots - \langle \vec{u}_{j-1}, \vec{v}_j \rangle \vec{u}_{j-1}$.

QR factorization

The Gram-Schmidt process represents a change of basis from the old basis $\overrightarrow{v}_1, \ldots, \overrightarrow{v}_m$ to a new orthonormal basis $\overrightarrow{u}_1, \ldots, \overrightarrow{u}_m$ of V. The QR factorization involves a change of basis matrix R such that

$$\begin{pmatrix} | & | & | & | & | \\ \overrightarrow{v}_1 & \cdot & \cdot & \cdot & \overrightarrow{v}_m \\ | & | & | & | & | \end{pmatrix} = \begin{pmatrix} | & | & | & | & | \\ \overrightarrow{u}_1 & \cdot & \cdot & \cdot & \overrightarrow{u}_m \\ | & | & | & | & | \end{pmatrix} R$$

i.e. $M = QR$;

where R is an upper triangle matrix with entries:
$r_{11} = ||\overrightarrow{v}_1||,\ r_{jj} = ||\overrightarrow{v}_j^{\perp}||$ (for $j = 2,\ldots,m$), and $r_{ij} = \langle \overrightarrow{u}_i, \overrightarrow{v}_j \rangle$ (for $i < j$) .

Example: Find the QR factorization of the matrix $M = \begin{pmatrix} 2 & 2 \\ 1 & 7 \\ -2 & -8 \end{pmatrix}$.

<u>Solution</u>: $Q = \dfrac{1}{3}\begin{pmatrix} 2 & -2 \\ 1 & 2 \\ -2 & -1 \end{pmatrix}$ and $R = \begin{pmatrix} 3 & 9 \\ 0 & 6 \end{pmatrix}$.