Lecture (15)

14) Analytic Continuation & Natural Barriers.

14.1) Analytic continuation

The process of extending the range of validity of a representation or more generally: extending the region of definition of an analytic f^n is Known as Analytic Continuation.

eg. $g(z) = \frac{1}{1-z}$ is the unique analytic continuation of $f(z) = \sum_{n=0}^{\infty} z^n$ outside the unit circle $|z| > 1$ where $f(z)$ diverges.

14.2) Th^m

If $f(z)$ & $g(z)$ are analytic in D & coincide in a sub-region or curve $D' \subset D$; then $f(z) = g(z)$ everywhere in D.

Corollary :- Let A, B and C are regions of analyticity of f, g and h respectively & $f(z) = g(z)$ in $A \cap B$; then $g(z)$ is the analytic continuation of $f(z)$ in the region B & likewise w/ $h(z) = g(z)$ in $B \cap C$ $\Rightarrow$ h is analytic continuation of g in C.

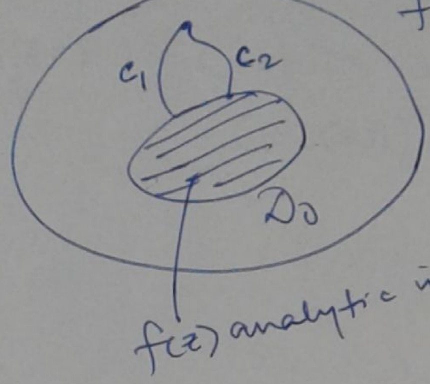
But this does not imply $h(z) = f(z)$ as $A \cap B \cap C$ may include a Branch pt. of a multi-valued f^n.

Def^n (14.1) Let D_1 and D_2 be 2 disjoint domains whose bdys share a common contour Γ. Let $f(z)$ be analytic in D_1 and continuous in $D_1 \cup \Gamma$ and $g(z)$ be analytic in D_2 & continuous in $D_2 \cup \Gamma$; and let $f(z) = g(z)$ on Γ. then

$$H(z) = \begin{cases} f(z) & ; z \in D_1 \\ f(z) = g(z) & ; z \in \Gamma \\ g(z) & ; z \in D_2 \end{cases}$$ is analytic in $D_1 \cup \Gamma \cup D_2$

14.3) Monodromy Theorem (Uniqueness of Analytic continuation).

Let D be a simply connected domain, $f(z)$ is analytic in some disk $D_0 \subset D$. If $f(z)$ can be analytically contd. to a pt. in D along 2 distinct smooth contours C_1 and C_2; then the result of each analytic continuation is the same & the f^n. is single valued; provided there are no singular pts enclosed by C_1 & C_2.



(th^m can be extended to the case where the reg^n. enclosed by C_1 & C_2 can have poles or essential singular pts)

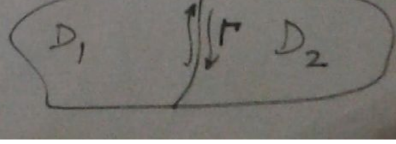
15) Natural Barrier (Bdy)

There are some types of non-isolated singularities that are in a sense, so serious that they prevent the analytic continuation of the f^n. in question.

$$\text{eg. } f(z) = \sum_{n=0}^{\infty} z^{2^n} \quad \text{across } |z| = 1$$

16) Mittag-Leffler expansions are certain suitable prescriptions for constructing meromorphic f^ns w/ prescribed principal parts in terms of suitable f^ns.

. We say $g(z)$ is the analytic continuation of $f(z)$.



Lecture (15) contd...

2) The Gamma function ($\Gamma(x)$)

2.1) Def^n $\Gamma(x) := \int_0^\infty t^{x-1} e^{-t}\, dt, \; x \in \mathbb{R}.$

$\Gamma(x)$ is uniformly convergent for $0 < a \le x \le b$

2.2) If $z \in \mathbb{C}$

i) $\Gamma(z) := \int_0^\infty t^{z-1} e^{-t}\, dt$ is uniformly convergent

for $\mathrm{Re}\, z \geq a > 0$

over a finite region.

ii) $\Gamma(z)$ is analytic for $\mathrm{Re}\, z > 0$.

2.3) for $x > 1$; $x \in \mathbb{R}$ (also true for $x \in \mathbb{C}$)

$\Gamma(x) = (x-1)\Gamma(x-1)$; $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$

2.4) $n \in \mathbb{I}^+$;

$\Gamma(n) = (n-1)! = (n-1)\Gamma(n-1)$

$\Gamma(1) = 1$;

2.5) $\log \Gamma(n) = (n - \frac{1}{2}) \log n - n + c + O(1)$; c = constant.

2.6) $\dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \displaystyle\int_0^\infty \frac{t^{y-1}}{(1+t)^{x+y}}\,dt$; $x, y > 0$

$y = 1-x$ gives.

$\Gamma(x)\Gamma(1-x) = \dfrac{\pi}{\sin \pi x}$; $0 < x < 1$. } True when $x \in \mathbb{C}$.

i.e. $\Gamma(z)\Gamma(1-z) = \dfrac{\pi}{\sin \pi z}$

$\mathrm{Re}\, z > 0$

3) Analytic Continuation of $\Gamma(z)$.

It can be shown that $\Gamma(z)$ is a regular f^n for $\mathrm{Re}\, z > 0$.

We now seek to extend $\Gamma(z)$ to the rest of the complex plane.

Q) When is a f^n called regular?

Ans) f is regular means f is analytic & single valued.

Recall the f^nal eqn. $\Gamma(z) = \dfrac{\Gamma(z+1)}{z}$

$\Gamma(z)$ is analytic when $\Gamma(z+1)$ is analytic for $z \neq 0$;

$\therefore \Gamma(z)$ can be "analytically continued" to $\mathrm{Re}\,(z+1) > 0$ i.e. $\mathrm{Re}\, z > -1$; $z \neq 0$.

Likewise; $\Gamma(z) = \dfrac{\Gamma(z+2)}{z(z+1)}$ & hence $\Gamma(z)$ can be analytically continued to $\mathrm{Re}\, z > -2$; $z \neq 0, -1$

& following likewise, $\Gamma(z)$ can be analytically continued to the entire complex plane minus the poles at $\{0, -1, -2, \ldots\}$.

We know that the analytic continuation is unique.

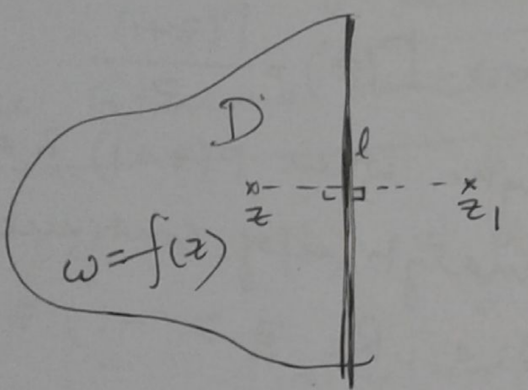
$\Gamma_m(z) = \dfrac{\Gamma(z+m)}{z(z+1)\cdots(z+m-1)}$ is the unique analytic continuation of $\Gamma(z)$ to $\mathrm{Re}\, z > -m$ minus $\{0, -1, -2, \ldots, -m+1\}$

4) The Principle of reflection.

Let $f(z)$ be an analytic f^n, regular in a region D intersected by the real axis, & real on the real axis. Then $f(z)$ takes conjugate values for conjugate values of z.

(Proof of above can be shown by analytic continuation).

5) Riemann - Schwarz principle of reflection.

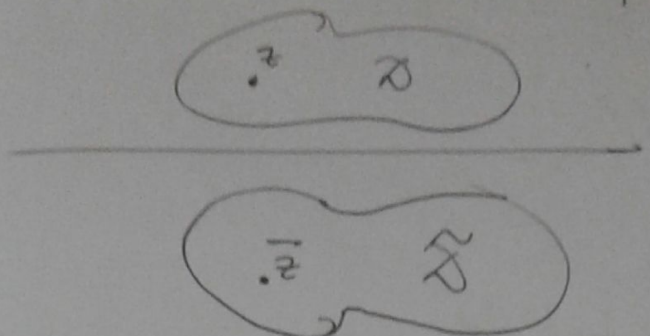


Let D be a region of the z-plane has a part of its bdy defined by the line segment l, and $\omega = f(z) = u(x,y) + iv(x,y)$ is analytic in D and continuous on l & s.t. as z describes l, ω describes a st. line λ in the ω-plane ((u,v) plane).

Let $z \in D$ & z_1 is the reflection in l & let ω_1 be the reflection of ω in λ. (this is analogous to the reality cond^n req'd on the bdy (real axis)).

Then $\omega_1 = \omega_1(z_1)$ is an analytic continuation of $\omega = f(z)$.

Let us illustrate this principle further.



Let $f(z)$ is analytic in $\mathcal{D}$ that lies in the UHP. $\tilde{\mathcal{D}}$ is the reflection of $\mathcal{D}$ w.r.t. the real axis then corresponding to every pt. $z \in \mathcal{D}$; the f^n $\tilde{f}(z) = \overline{f(\bar{z})}$ is analytic in $\tilde{\mathcal{D}}$.

Analytic Continuation :-

The reflection principle can be used as a method for analytic continuation as follows :-

Suppose $f(z)$ is continuous on the bdy $\{z \in \mathbb{C} \mid \operatorname{Im} z = 0\}$ & analytic in the UHP $\{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ s.t. $f(z)$ is real valued on the real axis

then $f(\bar{z}) = \overline{f(z)}$ (or $\tilde{f}(z) = \overline{f(\bar{z})}$) is the analytic continuation of $f(z)$ on the entire complex plane.

eg 5.1) $f(z) = \dfrac{1}{z+i}$ is analytic in UHP ($\operatorname{Im} z \geq 0$)

$\tilde{f} = \overline{f(\bar{z})} = \overline{\left(\dfrac{1}{\bar{z}+i}\right)} = \dfrac{1}{z-i}$ is analytic in LHP ($\operatorname{Im} z \leq 0$) (b/c its pole $z = i$ is in UHP).

Note $f(z)$ & $\tilde{f}(z)$ do not agree on the bdy $z = x + iy$, $y=0$

b/c $f(z) = \dfrac{1}{x+i} \neq \tilde{f} = \dfrac{1}{x-i}$ $\Rightarrow$ $\tilde{f}$ is NOT analytic continuation of f.

Also $f(z)$ is NOT real valued on real axis!

Ostrowski – Hadamard Gap thm

Let $0 < p_1 < p_2 < \dots$ be a sequence of integers s.t. for some $\lambda > 1$, $\forall j \in \mathbb{N}$

$$\frac{p_{j+1}}{p_j} > \lambda$$

Let $\{\alpha_j\}_{j\in\mathbb{N}}$ be a sequence of complex no.s such that

$$f(z) = \sum_{j\in\mathbb{N}} \alpha_j z^{p_j} \text{ has R.O.C.} = 1.$$

then no pt z w/ $|z|=1$ is a regular pt. for f i.e. f cannot be analytically extended from the open unit disc D to any larger open set including even a single pt. of ∂D.

eg $f(z) = \sum_{n=0}^{\infty} z^{2^n}$

$$= \sum_{k=0}^{\infty} a_{n_k} z^{n_k} \;;\; n_k = 2^k, \; a_{n_k} = 1$$

$$\text{R.O.C, } R = \frac{1}{\lim_{n\to\infty} \sqrt[n_k]{a_{n_k}}}$$

$$= \left(\lim_{n\to\infty} \sqrt[2^n]{1}\right)^{-1} = 1$$

$$\because \frac{n_{k+1}}{n_k} = \frac{2^{(k+1)}}{2^k} = 2 > 1 \;\forall k.$$

Hadamard's gap thm $\Rightarrow$ f has no analytic continuation outside $|z|<1$.