Last lecture: $P(Y \geq X) = \sum_{x=1}^{\infty} P(X = x, Y \geq X)$

$$= \dots = \sum_{x=1}^{\infty} P(X=x)\underbrace{P(Y \geq x)}_{?}$$

So what is $P(Y \geq x)$?

$$P(Y \geq x) = 1 - P(Y < x)$$

$$= 1 - \sum_{y=1}^{x-1} P(Y = y)$$

$$= 1 - \sum_{y=1}^{x-1} p(1-p)^{y-1} \quad \textbf{b/c } Y \sim \text{geom}_1(p)$$

$$\overset{m=y-1}{=} 1 - p\sum_{m=0}^{x-2} (1-p)^{m}$$

$$= 1 - p\,\frac{1-(1-p)^{x-1}}{1-(1-p)} \quad \text{b/c } \sum_{k=0}^{n-1} r^k = \frac{1-r^n}{1-r}; \text{ here } r = 1-p < 1$$

$$= (1-p)^{x-1}$$

So now back to our problem

$$P(Y \geq X) = \sum_{x=1}^{\infty} (1-p)^{x-1} p \, (1-p)^{x-1}$$

$$= p \sum_{x=1}^{\infty} (1-p)^{2(x-1)}$$

$$= \frac{p}{1-(1-p)^2}$$

b/c $\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$

here $r = (1-p)^2 < 1$

$$= \frac{p}{1-(1+p^2-2p)}$$

$$= \frac{1}{-(p-2)} = \frac{1}{(2-p)}$$

(a) $P(\underbrace{\min(X,Y)}_{\text{Let's call this } Z} > z) \overset{\text{why?}}{=} P(X > z, Y > z)$

$\overset{\text{independence}}{=} P(X > z) P(Y > z)$

$$P(Z > z) = (1-p)^{z-1}(1-p)^{z-1} = \{(1-p)^2\}^{z-1}$$

$\therefore Z \sim \text{geom}, (1 - (1-p)^2)$. ← think about this

(c) $$P(X+Y=z) = \sum_{x=1}^{z-1} P(X=x, X+Y=z) \quad \text{for any } z \geq 2$$

why ?

$$= \sum_{x=1}^{z-1} P(X=x, Y=z-x)$$

$$\stackrel{\text{indep.}}{=} \sum_{x=1}^{z-1} P(X=x)\,P(Y=z-x)$$

$$= \sum_{x=1}^{z-1} p(1-p)^{x-1}\, p(1-p)^{z-x-1}$$

$$= \ldots\ldots (z-1)p^2(1-p)^{z-2}$$

Simply use the sum of geometric series trick.

(d) $$P(Y=y \mid X+Y=z) \stackrel{\text{def}^n}{=} \frac{P(Y=y, X+Y=z)}{P(X+Y=z)}$$

why ?

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \equiv \frac{P(A,B)}{P(B)}$$

think of $A \equiv Y=y$
$B \equiv X+Y=z$!

$$= \frac{P(X=z-y, Y=y)}{P(X+Y=z)} \underset{\text{part (c)}}{\stackrel{\text{indep.}}{=}} \frac{P(X=z-y)\,P(Y=y)}{(z-1)p^2(1-p)^{z-2}}$$

$$= \ldots\ldots = \frac{1}{z-1}$$

Now let us attempt to workout problems using similar concepts but for continuous random variables!

Q) X and Y are independent RVs each w/ an exponential (λ) distribution. Find the pdf of
(i) $Z = X + Y$; and (ii) $W = Y - X^2$

Soln :- Clearly we expect $f_{XY}(x,y) = f_X(x) f_Y(y)$

$$\Rightarrow f_{XY}(x,y) = \begin{cases} \lambda e^{-\lambda x} \lambda e^{-\lambda y} & ; \text{ if } x, y \geq 0 \\ 0 & ; \text{ o.w.} \end{cases}$$

(i) $f_Z(z) = \frac{d}{dz} F_Z(z)$ where $F_Z(z) = P(Z \leq z)$

$$F_Z(z) = P(X + Y \leq z) = P(Y \leq -X + z)$$

Let $z \geq 0$; $F_Z(z) = P(Y \leq -X + z) = \int_0^{\infty} \int_0^{(-x+z)} f_{XY}(x,y)\, dy\, dx$

You should think of the outer integral as summing over all partitioning subsets (eg. $\sum_{x=1}^{\infty} P(\cdot,\cdot)$ of pg 1.

But for all $x > z$ $P(Y \leq -x+z)$

$$= P(Y \leq (\text{-ve no.})) = 0 .$$

(Why? think of the defn. of $\exp(\lambda)$ Dn.

$P(X \leq x)$

$= \int_0^x f_X(x)\,dx$

$= \int_0^x \lambda e^{-\lambda x}\,dx$

for all $x > 0$.

but 0 $\forall x < 0$)

$$\therefore F_Z(z) = \int_0^z \int_0^{(-x+z)} \lambda e^{-\lambda x} \lambda e^{-\lambda y}\,dy\,dx$$

$$= \begin{cases} 1 - e^{-\lambda z} - \lambda z e^{-\lambda z}; & \forall z \geq 0. \\ 0; & z < 0 \end{cases}$$

Consequently, $f_Z(z) = \begin{cases} 0; & z < 0 \\ \lambda^2 z e^{-\lambda z}; & z \geq 0 \end{cases}$

part (ii) :- Next Lecture!

Review Concepts :- for Continuous RVs

Marginal cdfs and pdfs

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy\,dx$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy \Rightarrow F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_X(x)\,dx$$