2) Let $\mathbb{V}$ be the set of all continuous functions $f:\mathbb{R}\to\mathbb{R}$, let the field of scalars be $\mathbb{R}$, and let the operations be usually defined.
HW: Verify that the above indeed constitutes a vector space!

3) Let $\mathbb{V}$ be the set of all continuous functions $f:\mathbb{R}\to\mathbb{R}$ that satisfy the equation $f''=-f$. Can you think of any function that satisfies this property? Cosine, Sine? Let the field of scalars be $\mathbb{R}$. The operations are defined in the usual manner. *Hint: Suppose* $f_1, f_2 \in \mathbb{V}, \quad c \in \mathbb{R};$ *then* $(f_1+f_2)''=f_1''+f_2''=-f_1-f_2=-(f_1+f_2)$*; and* $(cf_1)''=cf_1''=c(-f_1)=-(cf_1)$. *Are these results consistent with the definition of the vector space? Also check whether all axioms are compliant?*

4) Transactions on an accounting system (bank). Think about this for now. We will return to this example soon!

Linear independence of vectors

Do they all look alike? Do they upon each rely?

Definition (<u>Linearly dependent vectors</u>):
Let $\mathscr{V}$ be a vector space and $\mathscr{X}\subset\mathscr{V}$ be a non-empty subset. Then $\mathscr{X}$ is ***<u>linearly dependent</u>*** if there are distinct vectors $v_1, v_2, \ldots, v_k \in \mathscr{X}$, and scalars $c_1, c_2, \ldots, c_k$ (not all of them zero) , s.t. $c_1v_1+c_2v_2+\ldots+c_kv_k=0$.

This is equivalent to saying that *at least one of the vectors* v_i *can be expressed as a linear combination of the others.*

$$v_i=\sum_{j\neq i}-\left(\frac{c_j}{c_i}\right)v_j$$

Definition (<u>Linearly independent vectors</u>):
A subset which is not linearly dependent is said to be *linearly independent.* Thus a set of distinct vectors $\{v_1, v_2, \ldots, v_k\}$ is linearly independent if and only if an equation of the form $c_1v_1+c_2v_2+\ldots+c_kv_k=0$ always implies that $c_1=c_2=\ldots=c_k=0$.

Geometrical interpretation of linear dependence

So what does this look like? What does it manifest?

Let V_1, V_2, V_3 be the vectors in 3D-Euclidean space $\mathbb{R}^3$ with a common origin. If these vectors form a *linearly dependent* set, then one of them, say V_1, can be expressed as a linear combination of the other two: $V_1 = aV_2 + bV_3$. This implies, by the parallelogram law, that the three vectors are **co-planar.**

In fact, **linearly dependent set of vectors with common origin $\Longleftrightarrow$ co-planar.**

Can you think of a similar interpretation of vectors in $\mathbb{R}^2$?

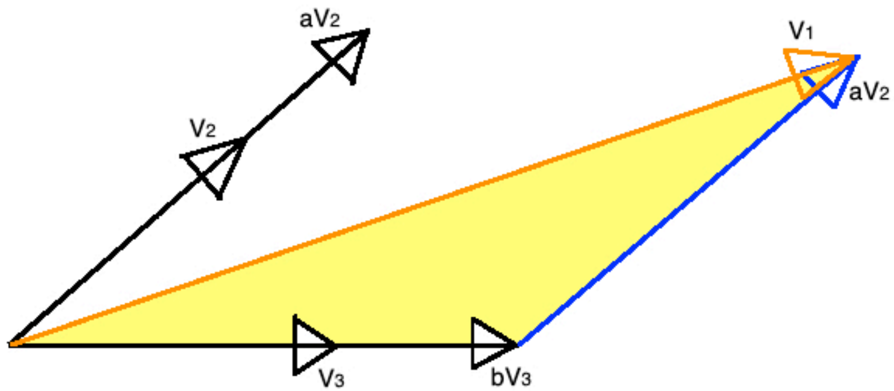


Fig. 1: Linear dependence of vectors is equivalent to coplanar geometry.

Basis of a vector space

What holds it all together? What is the skeleton on which it stands?

Definition (Basis): Let $\mathbb{X}$ be a non-empty subset of a vector space $\mathscr{V}$. Then $\mathbb{X}$ is called a *basis* of $\mathscr{V}$ if both the following are true:

i. $\mathbb{X}$ is *linearly independent*,

ii. $\mathbb{X}$ generates $\mathscr{V}$ (i.e. $\mathbb{X}$ *spans* $\mathscr{V}$).

What is the meaning of "*spans*"? Technically, it means that every element (vector) in the space $\mathscr{V}$ can be expressed as a linear combination of the elements of the set $\mathbb{X}$.

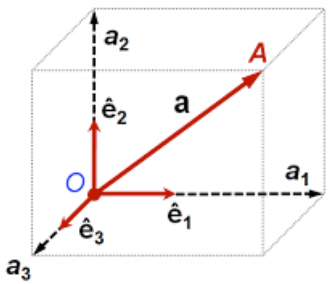
Examples of bases

Can you give me some examples?

1. *Basis of* $\mathbb{R}^n$: $\mathbf{e_1} = \begin{pmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix}, \ \mathbf{e_2} = \begin{pmatrix} 0 \\ 1 \\ \cdot \\ \cdot \\ 0 \end{pmatrix}, \ \dots, \ \mathbf{e_n} = \begin{pmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 1 \end{pmatrix}$ form a basis of $\mathbb{R}^n$ because they (i) are *linearly independent* (by inspection),

   and (ii) *span* $\mathbb{R}^n$ owing to the fact that $c_1\mathbf{e_1} + c_1\mathbf{e_2} + \cdots c_n\mathbf{e_n} = \begin{pmatrix} c_1 \\ c_2 \\ \cdot \\ \cdot \\ c_n \end{pmatrix}$ generates any vector in $\mathbb{R}^n$ depending on the values of

   $c_i \quad \forall i = 1,2,\ldots,n$.



2. Let $\mathbb{P}_n$ be a vector space of all polynomial functions of degree n or less. The basis of $\mathbb{P}_n$ is $\{1, \; x, \; x^2, \; \ldots, \; x^n\}$, the set of monomials. (This is not a unique basis set because $\{p_0(x), \; p_1(x), \; \ldots, \; p_n(x)\}$ also forms a basis where $p_i(x)$ is a polynomial in $\mathbb{P}_n$ of degree i .)

3. Let $\mathbb{M}_{m\times n}(\mathbb{F})$ denote the set of $m \times n$ matrices with entries in $\mathbb{F}$. Then $\mathbb{M}_{m\times n}(\mathbb{F})$ is a vector space over $\mathbb{F}$. Vector addition is just matrix addition and scalar multiplication is defined in the obvious way (by multiplying each entry of the matrix by the same scalar). The zero vector is just the zero matrix. One possible choice of basis is the matrices with a single entry equal to 1 and all other entries 0. (We will study the vector space of matrices in more detail soon in the subsequent lectures!)

Properties of bases

What the bleep do we know?

1. **Must every vector space have a basis?**

   Ans: Every non-zero, finitely generated vector space has a basis!

2. **Does a vector space have a unique basis?**

   Ans: Usually a vector space will have many bases. Eg., the vector space $\mathbb{R}^2$ *has the basis* $\left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right\}$

   as well as the standard basis $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$.

3. **What is the dimension of a vector space?**

   Ans: $dim(\mathscr{V}) =$ *no. of elements (vectors) in the basis (basis set) . Can you think of a vector space whose dimension is infinite?*

We will return to bases & vector spaces again, shortly, after we take a detour into the world of matrices!