Agenda items

3/1/2022

Module 2
Lec. Set 2

- → Eigenvalues (evs) and Eigenvectors (EVs) of a matrix
- → Meaning of evs and EVs
- → Diagonalizable matrices & Similarity transformations
- → Analytical (pen-paper) method of finding evs.
- → Computational method of finding evs. of a matrix (power method, etc)

Defⁿ (evs and EVs):

may be $\mathbb{R}$ or $\mathbb{C}$

Let $A \in M_{n\times n}(\mathbb{F})$

$\vec{x}$ is an EV of A if

$\vec{x} \in \mathbb{F}^n$

$$A\vec{x} = \lambda\vec{x}$$

constant (either $\mathbb{R}$ or $\mathbb{C}$)

cannot be $\vec{0}$!

λ is an ev of A associated w/ the EV $\vec{x}$!

Meaning of the eq. $A\vec{x} = \lambda\vec{x}$

(i) Algebraic meaning

$A\vec{x} = \lambda\vec{x}$ can also be written as

$$(A - \lambda I)\vec{x} = \vec{0}$$

recall we had left out $\vec{0}$ from the def^n of EV!

i.e. the (EV $\vec{x}$ + $\vec{0}$) form the null sp. of the matrix $(A - \lambda I)$

this is a subspace of $(A-\lambda I)$ that has a spl! name "EIGENSPACE" of λ w.r.t. A!

Why?

$$\underbrace{\lambda\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{\lambda\vec{x}} \equiv \underbrace{\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{\lambda I\vec{x}}$$

in search of meaning of $A\vec{x} = \lambda\vec{x}$ . . . .

Consider an example

$$A = \begin{pmatrix} 2 & -1 \\ 2 & 4 \end{pmatrix}$$

$$(A - \lambda I) = \begin{pmatrix} 2-\lambda & -1 \\ 2 & 4-\lambda \end{pmatrix}$$

Solving $A\vec{x} = \lambda\vec{x}$ is equivalent to solving the system of linear eqs.

$$\begin{pmatrix} 2-\lambda & -1 \\ 2 & 4-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} (2-\lambda)x_1 - x_2 = 0 \\ 2x_1 + (4-\lambda)x_2 = 0 \end{cases}$$

b/c EV can NOT be 0!

When does this system of linear eqns. have a non-trivial soln. ??

Ans:-

$\mathrm{Ker}(A-\lambda I) \neq \{\vec{0}\}$

this is true only if

$(A-\lambda I)$ is non-invertible

characteristic eqn.

i.e. $\boxed{\det(A-\lambda I) \equiv |A-\lambda I| = 0}$

i.e. $\begin{vmatrix} 2-\lambda & -1 \\ 2 & 4-\lambda \end{vmatrix} = 0 \Rightarrow \begin{cases} \lambda_1 = 3+i \\ \lambda_2 = 3-i \end{cases}$

EV w.r.t. λ_1: $x_1 = k(-1+i)/2$
$x_2 = k$

some arb. const.

Various features of an invertible matrix

$B \in M_{n\times n}$

1) B is invertible
2) $B\vec{x} = \vec{b}$ has a unique soln. $\vec{x}$ $\forall \vec{b} \in \mathbb{R}^n$
3) $\mathrm{rref}(B) = I_n$
4) $\mathrm{rank}(B) = n$
5) $\mathrm{im}(B) = \mathbb{R}^n$
6) $\mathrm{Ker}(B) = \{\vec{0}\}$

all the above statements are equivalent

A slight digression - - - -

Q) Why $\text{null}(B) = \{0\} \iff B$ is ~~not~~ invertible?

finite dim. vector sp.

Ans) $T : U \longrightarrow V$ is invertible if and only if T is $\left\{\begin{array}{l}\text{one to one} \\ \& \\ \text{onto}\end{array}\right.$

by inspecting im & ker of T

$$\Rightarrow \dim(U) = \dim(V)$$

Rank nullity thm :- $\underbrace{\text{null}(T)}_{\{0\}} + \underbrace{\text{rank}(T)}_{\dim(V)} = \dim(U)$

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HW/exercise problem

Q) Consider $A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 0 \end{pmatrix}$

i) Find the characteristic polynomial.

ii) Find the evs of A.

iii) Find the EVs of A.

Ans :- evs : $-1, 2, 3$

EVs : $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

ii) Geometrical meaning of

$A\vec{x} = \lambda \vec{x}$ ....

when λ is real ...

$A\vec{x}$ is $\parallel^{lel}$ to $\vec{x}$

i.e. the "EV" $\vec{x}$ either gets stretched or compressed longitudinally when acted upon by the matrix A.

Coming Soon !

* Diagonalizable matrices
* Similarity transformation
* Application of evs & EVs in solutions to ODEs