# Midsem Rubric

**Answer 1:(10 marks)** $X$ and $Y$ are independent random variables each with an exponential distribution $\exp(\mu_1)$ and $\exp(\mu_2)$, respectively. Here, $\frac{1}{\mu_1} = \frac{1}{\mu_2} = 100 = \frac{1}{\lambda}(say)$.
To compute the probability of dual engine flame out to be more than 75 flying hours, we have to compute $P(X+Y > z)$ where $z = 75$.

$$P(X+Y > z) = 1 - P(X+Y \leq z). \qquad \text{(2 marks)}$$

First we will compute

$$\begin{aligned}
P(X+Y \leq z) &= \int_0^z P(X \leq z-y|Y=y) f_Y(y)dy \\
&= \int_0^z P(X \leq z-y) f_Y(y)dy \\
&= \int_0^z \left( \int_0^{z-y} f_X(x)dx \right) f_Y(y)dy \\
&= \int_0^z \left( \int_0^{z-y} \lambda \exp(-\lambda x)dx \right) \lambda \exp(-\lambda y)dy \qquad \text{(5 marks)} \\
&= \int_0^z \lambda(1 - \exp(-\lambda(z-y))) \exp(-\lambda y)dy \\
&= 1 - \exp(-\lambda z) - \lambda z \exp(-\lambda z). \\
F_{X+Y}(z) &= 1 - \exp(-\lambda z) - \lambda z \exp(-\lambda z).
\end{aligned}$$

$$P(X+Y > 75) = 1-F(75) = 1-(1-\exp(-\lambda z)-\lambda z.\exp(-\lambda z))|_{(z=75)} = 0.8266 \quad \text{(3 marks)}$$

**Answer 2 (10 marks):**Let $T_k$ = number of cells in generation k. Then $T_k = X_1 + X_2 + \cdots + X_{T_{k-1}}$ where $X_1, \cdots, X_{T_{k-1}}$ are the of progeny of the first, second, $\cdots$, $T_{k-1}^{th}$ cell in generation $k-1$. So,

$$E[T_k] = E_{T_{k-1}}[E[T_k \mid T_{k-1}]] \qquad \text{(2 marks)}$$

$$= E[T_k \mid T_{k-1} = 1]P(T_{k-1} = 1) + E[T_k \mid T_{k-1} = 2]P(T_{k-1} = 2) + \cdots$$
$$= E[T_k = X_1]P(T_{k-1} = 1) + E[T_k = X_1 + X_2]P(T_{k-1} = 2) + \cdots \qquad \text{(2 marks)}$$

$$= \mu P(T_{k-1} = 1) + 2\mu P(T_{k-1} = 2) + \cdots$$
$$= \mu \sum_{n=1}^{n=\infty} nP(T_{k-1} = n)$$
$$= \mu E[T_{k-1}]. \qquad \text{(3 marks)}$$

Hence, with the initial condition $E[T_0] = 1$, we will get $E[T_k] = \mu^k E[T_0] = \mu^k$.

So,

$$\text{Total tumor cells over n generations} = \sum_{k=0}^{k=n} E[T_k] \qquad \text{(1 mark)}$$
$$= 1 + \mu + \mu^2 + \cdots + \mu^n = \frac{1 - \mu^{n+1}}{1 - \mu}.$$

Clearly, this diverges whenever $\mu \geq 1$. To halt the growth, we need $\mu < 1$ (assuming that previous generation does not die at every step, if you assume it dies, then $\mu = 1$ also works). In this case, (1 mark)

$$\lim_{n\to\infty} \sum_{k=0}^{k=n} E[T_k] = \frac{1}{1-\mu}. \qquad \text{(1 mark)}$$

**Answer 3 (10 marks):**

a) **(8 marks)** Let $X_i$ be a random variable denoting the number of balls in the $i^{th}$ bin. Where it is 1, when there is no ball in $i^{th}$ bin and 0 with at least 1 ball in the bin. So,

$$X_i = \begin{cases} 1, & \text{when } i^{th} \text{ bin is empty} \\ 0, & \text{otherwise.} \end{cases} \qquad \text{(2 marks)}$$

We need to find the probability distribution of this random variable for finding the probability. This is

$$P(X_i = j) = \begin{cases} \left(\frac{b-1}{b}\right)^m, & j = 1 \\ 1 - P(X_i = 1), & \text{otherwise.} \end{cases} \qquad \text{(3 marks)}$$

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**Note:** You can also directly write that this is Bernouli random variable with $p = \left(\frac{b-1}{b}\right)^m$. If the value of $p$ is not right, you do not get credit.

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We need to find expectation of the random variable $Y$ where $Y = X_1 + \cdots + X_b$. Clearly, $Y$ denotes the number of empty bins. It is clear that $X_i$ should be independent and identically distributed. So,

$$E[Y] = bE[X] = b\left(\frac{b-1}{b}\right)^m. \qquad \text{(3 marks)}$$

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**Note:**

a) Since $Y = X_1 + \cdots + X_b$ and $X_i \sim Bin(1, p)$ where $p = \left(\frac{b-1}{b}\right)^m$, you can directly write $Y \sim Bin(b, p)$ and use the formula for expectation of Binomial, i.e. $E[Y] = bp = b\left(\frac{b-1}{b}\right)^m$. If your value of $n$ or $p$ in formula is not correct, you loose full credit.

b) It is possible that you have swapped $1$ and $0$ when you have defined your random variable $X_i$, you will still get full credit.

c) If you have calculated probability and expectation for only $1$ or $0$ ball allowed in a bin, you do not get any credit. You were expected to calculate it in the case that any number of balls are allowed to be in a bin (eg all the balls could be in a bin).

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b) **(2 marks)** To evaluate the limit, we can proceed from $E[Y]$. Since, $b$ is very large, we can make the approximation $\left(1 - \frac{1}{b}\right)^b \approx e^{-1}$. Thus, $E[Y] = be^{-\frac{m}{b}}$ and hence $E[Y] \to \infty$ as $b \to \infty$. Note that for large $b$, we can use the Taylor approximation to the exponential, i.e.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$$

Then, $be^{-\frac{m}{b}} = b\left(1 - \frac{m}{b} + O\left(\frac{1}{b}\right)\right) \approx b(1 - \frac{m}{b}) = b - m$.

**Alternatively,**

$E[Y] \geq b - m$ since we at least have $b - m$ empty bins even if all $m$ balls fall in different bins. In the limit $b \to \infty$, it is clear that $E[Y] \to \infty$. In fact, $E[Y] \approx b - m$, to see this, see the method below.

**Alternatively,**

We can also proceed from

$$
\begin{aligned}
E[Y] = b\left(\frac{b-1}{b}\right)^m =&(b-1)\left(1-\frac{1}{b}\right)^{m-1} \\
=&(b-1)\left(\binom{m}{0} - \binom{m}{1}\left(\frac{1}{b}\right) + \binom{m}{2}\left(\frac{1}{b}\right)^2 + \cdots \text{(higher order terms)}\right) \\
=&(b-1)\left(1 - \left(\frac{m}{b}\right) + \frac{m(m-1)}{2}\left(\frac{1}{b}\right)^2 + \cdots \text{(higher order terms)}\right) \\
=&b - m + O\left(\frac{1}{b}\right) \approx b - m, \quad \text{for large } b.
\end{aligned}
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**Note:** You only get credit for this part if you are able to explain why $E[Y] \to \infty$ when $b \to \infty$ or alternatively if you are able to sufficiently show that $E[Y] \approx b - m$ when $1 < m \ll b$.