LESLIE MATRIX MODEL OF DEMOGRAPHIC TRENDS (AN APPLICATION OF EIGENVALUES AND EIGENVECTORS)

Goal: Demographic analysis using the Leslie matrix model

Getting Started :

- Open a new Jupyter notebook and save it as name_Stableage.ipynb.
- **Useful Python commands:** `eye, eigvec, sum, abs, argmax, for,append, zeros,inv, det, for, while ...end, if-elseif`

1. Conceptual Overview

This project employs the Leslie Matrix to analyze population growth and estimates age distribution within a population over time. The model was originally formulated by P. H. Leslie in 1945. This mathematical framework has since found application in population dynamics of organisms such as brook trout, rabbits, lice, beetles, pine trees, buttercups, killer whales, and humans. In this project, we will make the following assumptions

- The population consists exclusively of males and females and is categorized into three age groups.
- The average survival rate for males and females in each age group is known, i.e. number of individuals who remain in the same age group is known to us.
- The reproduction (fecundity) rate for each female belonging to each age group is known and is further categorized into two categories based on male child and female child reproduction rates of females.
- The migration rate for males and females from one age group to the next age group is also known.
- The initial age distribution is known.

The population can be broadly divided into three distinctive age groups. The initial age category, designated as Class 1, typically includes individuals from birth to 18 years. Class 2 represents the adult age group comprising individuals from 19 to 40 years. This is the stage when most people are considered to be in their prime working and reproductive abilities. Finally, class 3 includes seniors whose age exceed 40 years.

2. Mathematical Model

Let us envision a scenario where we possess information regarding the population count within three distinct age groups, comprising both male and female individuals at a specific time $t = t_k$.

Generally, at time $t = t_k$, we denote the age distribution vector by $\mathbf{Z^{(k)}}$ which consists of entries denoting the population count of females and males within the first, second, and third age categories at any time t. In this context, we can represent the initial age composition by $\mathbf{Z^{(0)}}$.

Over the course of time, alterations in the population composition within the age groups happen as a consequence of three fundamental biological mechanisms: birth, mortality, and the process of ageing.
Our investigation of the demographic will occur at distinct one-year intervals, which we shall designate as $t_0, t_1, t_2, t_3, \ldots$. The dynamics due to birth, growth, and mortality between consecutive observation points can be effectively studied based on parameters such as the "average" reproduction rate, the "average" migration rate, and the "net" survival rate.
Let F_i^f be the average number of females born to a single female in the i^{th} age group and F_i^m be the average number of males born to a single female in the i^{th} age group $(i = 1, 2, 3)$. F_i is the average reproduction rate of a single female in the i^{th} age group. Let P_i be the fraction of males in the i^{th} age class that survive in the same age group and remain in the same age group. Similarly, P_i' is the fraction of females in the i^{th} age group that survive in the same age group and remain in the same age group. Also, assume $M_{i,i+1}$ and $M'_{i,i+1}$ be the fraction of males and females in the i^{th} age group that survive in the same age group and migrate to the $(i+1)^{th}$ age group.

By definition, $F_i^f, F_i^m \geq 0$ since the number of offspring produced cannot be negative. We assume that females only in the second age group can reproduces which implies

$$F_1^m = F_3^m = 0 \quad \text{and} \quad F_1^f = F_3^f = 0.$$

Also, $0 \leq P_i, P_i' \leq 1$ for $i = 1, 2, 3$, since we assume that some of the individuals (both males and females) must survive and remain in the same group. Similarly, $0 \leq M_{i,i+1}, M'_{i,i+1} \leq 1$ for $i = 1, 2,$.

We denote the age distribution at time $t = t_k$ by $\mathbf{Z^{(k)}}$ and at time $t = t_{k-1}$ by $\mathbf{Z^{(k-1)}}$ whose components will be all male and female populations at time $t = t_k, t_{k-1}$.

The population model can be written in the mathematical form as follows:

$$\mathbf{Z^{(k)}} = \mathbf{L} * \mathbf{Z^{(k-1)}}.$$

The matrix L is generally called the Leslie matrix, and the population can be regarded to be in stable age distribution if

$$\mathbf{Z^{(k)}} = \mathbf{L} * \mathbf{Z^{(k-1)}} = \mathbf{Z^{(k-1)}}. \tag{2.1}$$

Observe that the population at time $t = t_k$ can be written as:

$$\mathbf{Z^{(k)}} = \mathbf{L} * \mathbf{Z^{(k-1)}} = \mathbf{L^2} * \mathbf{Z^{(k-2)}} = \mathbf{L^3} * \mathbf{Z^{(k-3)}} = \cdots = \mathbf{L^k} * \mathbf{Z^{(0)}}.$$

3. Case Study: Population Growth

The population (both males and females) is classified into three distinct age cohorts: those aged 18 years or younger, individuals ranging from 19 to 40 years old, and seniors aged 41 years and above.

The focal point of this project lies in the development of a Leslie matrix model, a stage-structured framework, tailored to this population. To construct this matrix, critical data, including age-specific annual survival rates and the number of male and female infants born each year per female, will be harnessed and meticulously integrated. The ages, annual survival rates, and number of male and female infants per year by each female are provided in the following table.

Population groups	Age	Category	Rate of survival per year		No. of newborns per female	
			Stayed in same age group	Migrated to next age group	Male	Female
Juveniles	$\leq$ 18 Years	Females	$P'_1 = \frac{389}{1000}$	$M'_{1,2} = \frac{23}{100}$	-	-
		Males	$P_1 = \frac{7}{20}$	$M_{1,2} = \frac{4}{25}$	-	-
Adults	19-40 Years	Females	$P'_2 = \frac{151}{611}$	$M'_{2,3} = \frac{2}{5}$	$F_2^m = 1$	$F_2^f = 2$
		Males	$P_2 = \frac{11}{50}$	$M_{2,3} = \frac{37}{100}$	-	-
Seniors	$\geq$ 41 Years	Females	$P'_3 = \frac{7}{10}$	0	-	-
		Males	$P_3 = \frac{457}{1000}$	0	-	-

Table 3.1: Data for population

Use the data provided above to answer the following questions.

Question 1. *Express the population model using the data in terms of an equation*

$$\mathbf{Z^{(k+1)}} = L\mathbf{Z^{(k)}}. \tag{3.1}$$

Also, write the Leslie matrix for this population model.

Question 2. *Use a* `for` *loop to iterate the above Leslie equation (3.1) 50 times with initial age distribution as:*

Population groups	***Age***	***Category***	***No. of individuals***
Juveniles	$\leq$ *18 Years*	*Females*	*50*
		Males	*50*
Adults	*19-40 Years*	*Females*	*50*
		Males	*50*
Seniors	$\geq$ *41 Years*	*Females*	*50*
		Males	*50*

Table 3.2: Initial age distribution

Also, find the percentage of the juvenile female population in the 50^{th} *year. Use* `PYTHON` *to plot the natural logarithm of each individual (males & females) of each age group versus time (in years). What is the eventual fate of this population?*

Question 3. *Write a program in* `PYTHON` *to find the basis of the nullspace of the matrix* $L - I$*, where* L *is the Leslie matrix obtained in Question (1) and* I *denotes the identity matrix.*

Question 4. *Deduce a connection between the basis of nullspace obtained in Question (3) and the age distribution of the* 50^{th} *year obtained in Question (2).*

Question 5. *Write a program in* `PYTHON` *to compute the eigenvalues and eigenvectors of the matrix* L*. What is the dominant eigenvalue and the corresponding eigenvector?*

Question 6. *Write a program in* ***PYTHON*** *to solve the system using the Gauss elimination method and hence solve the system*

$$LX = B, \quad \text{where} \quad B = (0.20043, 0.04111, 0.02801, 0.42645, 0.13028, 0.17370)^T.$$

Deduce the connection between the answer to Question (5), the solution to equation (2.1) and the solution of this question.

Question 7. *Now reconsider the data set given in Table 3.3 and find the condition on parameters 'a' and 'b' such that in the long run the population will diminish. Write a program in* ***PYTHON*** *to plot the region for these parameters that satisfy this condition.*

Population groups	***Age***	***Category***	***Rate of survival per year***		***No. of newborns per female***	
			Stayed in same age group	*Migrated to next age group*	*Male*	*Female*
Juveniles	$\leq$ *18 Years*	*Females*	$P'_1 = a$	$M'_{1,2} = \frac{23}{100}$	-	-
		Males	$P_1 = \frac{7}{20}$	$M_{1,2} = \frac{4}{25}$	-	-
Adults	*19-40 Years*	*Females*	$P'_2 = b$	$M'_{2,3} = \frac{2}{5}$	$F_2^m = 1$	$F_2^f = 2$
		Males	$P_2 = \frac{11}{50}$	$M_{2,3} = \frac{37}{100}$	-	-
Seniors	$\geq$ *41 Years*	*Females*	$P'_3 = \frac{7}{10}$	*0*	-	-
		Males	$P_3 = \frac{457}{1000}$	*0*	-	-

Table 3.3: Data for population