Matrix and vector norms

Let $\vec{x} \in \mathbb{R}^n$ i.e. $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

A vector norm on $\mathbb{R}^n$ is a f^n $\|\cdot\| : \mathbb{R}^n \longrightarrow \mathbb{R}$

w/ the following properties

(1) $\|\vec{x}\| \geq 0 \quad \forall \vec{x}$

(2) $\|\vec{x}\| = 0 \iff \vec{x} = \vec{0}$

(3) $\|\alpha \vec{x}\| = |\alpha| \|\vec{x}\| \quad \forall \alpha \in \mathbb{R}, \forall x \in \mathbb{R}^n$

(4) $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\| \quad \forall \vec{x}, \vec{y} \in \mathbb{R}^n$

There are <u>many types</u> of norms

(1) $\|\vec{x}\|_2 = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$

(2) $\|\vec{x}\|_\infty = \max_{1 \leq i \leq n} |x_i|$

* All norms are equivalent!

Cauchy-Schwarz inequality

$$\langle \vec{x}, \vec{y} \rangle \equiv \vec{x}^T \vec{y} = \sum_{i=1}^{n} x_i y_i \leq \|\vec{x}\|_2 \|\vec{y}\|_2$$

What is the use of norms?

(1) To measure distance b/n 2 pts. in space.

(2) Convergence of sequences (eg. analysis/monitoring of error of iterative methods)

A sequence $\{\vec{x}_{(k)}\}_{k=1}^{\infty}$ of vectors in $\mathbb{R}^n$ is said to converge to $\vec{x}$ w.r.t. the norm $\|\cdot\|$ if for any small $\epsilon > 0$, $\exists\ N(\epsilon)$ s.t. $\|\vec{x}_{(k)} - \vec{x}\| < \epsilon \ \forall\ k \geq N(\epsilon)$

<u>th^m</u> $\{\vec{x}_{(k)}\} \to \vec{x}$ in $\mathbb{R}^n$ w.r.t. $\|\cdot\|_\infty \iff \lim_{k\to\infty} x_{i(k)} = x_i$ for each $i = 1, 2, \ldots, n$

<u>th^m</u> for each $\vec{x} \in \mathbb{R}^n$

$$\|\vec{x}\|_\infty \leq \|x\|_2 \leq \sqrt{n}\,\|\vec{x}\|_\infty$$

X

Matrix norms

Let $A \in M_{n\times n}$; $\|\cdot\|$ is a f^n that maps A in $M_{n\times n}$ to a real-valued no.

properties of matrix norms

(1) $\|A\| \geq 0$

(2) $\|A\| = 0 \iff A$ is a 'O' matrix
w/ all entries $= 0$.

(3) $\|\alpha A\| = |\alpha| \, \|A\|$

(4) $\|A + B\| \leq \|A\| + \|B\|$

(5) $\|AB\| \leq \|A\| \, \|B\|$

Thm (Induced matrix norm)

If $\|\cdot\|$ is a vector norm on $\mathbb{R}^n$ then

$$\|A\| := \max_{\|\vec{x}\|=1} \|A\vec{x}\| \text{ is a matrix norm.}$$

$$\max_{\|\vec{x}\|=1} \|A\vec{x}\| \rightarrow \max_{\vec{z} \neq \vec{0}} \left\| A\left(\frac{\vec{z}}{\|\vec{z}\|}\right) \right\| = \max_{\vec{z} \neq 0} \frac{\|A\vec{z}\|}{\|\vec{z}\|}$$

<u>Th^m</u> ($\|\cdot\|_\infty$ matrix norm).

def $A = (a_{ij}) \in \mathbb{M}_{n\times n}$;

$$\|A\|_\infty = \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|$$

eg. $A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & -1 \\ 5 & -1 & 1 \end{pmatrix}$

$\longrightarrow \sum_{j=1}^{3} |a_{1j}| = |1| + |2| + |-1| = 4$

$\longrightarrow \sum_{j=1}^{3} |a_{2j}| = 4$

$\longrightarrow \sum_{j=1}^{3} |a_{3j}| = 7$

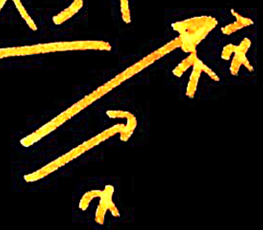
$\therefore \|A\|_\infty = \max\{4, 4, 7\}$
$= 7$

Eigenvalues (ev) & Eigenvectors (EV)

$$A\vec{x} = \lambda\vec{x}$$

when $\lambda \in \mathbb{R}$

$\lambda > 1$



$-1 < \lambda < 0$

EV of A is such a vector which when transformed by A → Shrinks/expands itself either in the dirn of itself or in the opposite dirn.

Q) How to find evs & EVs of a matrix?

Ans) Set Characteristic polynomial of A

$$p(\lambda) = |A - \lambda I| = 0.$$

Why?

Consider $A\vec{x} = \lambda\vec{x} \Rightarrow (A - \lambda I)\vec{x} = \vec{0}$

since $\vec{x} \neq \vec{0}$ (else we have triviality)

Now b/c $\underbrace{\|(A - \lambda I)\vec{x}\| \leq \|A - \lambda I\|}\, \|\vec{x}\| \quad \forall\, \vec{x} \neq \vec{0}$

this will get us no-where b/c determinants and norms have diff. meaning.

We are solving

$$\underbrace{(A - \lambda I)\vec{x} = \vec{0}}$$

this has a non-zero soln. for $\vec{x}$ if and only if $\det(A - \lambda I) \equiv |A - \lambda I| = 0$

this follows from an important thm that states "a linear sys. of eqns $B\vec{X} = 0$ w/ n eqns. & n unknowns has a non-trivial soln. if & only if $\det(B) = 0$"

Q) Find the EVs and evs of $A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$?

Ans :- evs $= \{0, 2\}$

EVs $= \left\{ \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \right\}$