Spectral radius of a matrix

$$\rho(A) := \max_{1 \le i \le n} |\lambda_i| \; ; \text{ where } \lambda_1, \ldots, \lambda_n \text{ are evs of } A \in M_{n\times n}$$

* spectral radius is closely related to the norm of a matrix!

Th^m :- Let $A \in M_{n \times n}$;

(i) $\|A\|_2 = \sqrt{\rho(A^{t}A)}$

(ii) $\rho(A) \le \|A\|$ where $\|\cdot\|$ is an induced matrix norm.

eg. $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix}$. Find $\|A\|_2$.

$B = A^{t}A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 2 & -1 \\ 2 & 6 & 4 \\ -1 & 4 & 5 \end{pmatrix}$

Now $|B - \lambda I| = -\lambda^3 + 14\lambda^2 - 42\lambda$
$= 0$

$\Rightarrow \lambda = 0;\ 7 \pm \sqrt{7}$

$\|A\|_2 = \sqrt{\rho(A^{t}A)} = \sqrt{7+\sqrt{7}} = 3.106$

A simple matrix decomposition

Consider any matrix, say

$$A = \begin{pmatrix} 2 & -1 & 5 \\ 0 & 1 & -2 \\ 1 & 5 & -1 \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}} + \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 5 & 0 \end{pmatrix}} + \underbrace{\begin{pmatrix} 0 & -1 & 5 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}}$$

$$= D + L + U$$

Can always do this splitting!

L: (strictly lower triangular)

U: (strictly upper triangular)

$$= D - (-L) - (-U)$$

Iterative Schemes to solve systems of linear eqns.

Jacobi iterative method

eg. Let us consider the system of linear eqns

$$E_1: \quad 10x_1 - x_2 + 2x_3 = 6$$

$$E_2: \quad -x_1 + 11x_2 - x_3 + 3x_4 = 25$$

$$E_3: \quad 2x_1 - x_2 + 10x_3 - x_4 = -11$$

$$E_4: \quad 3x_2 - x_3 + 8x_4 = 15$$

which has a unique soln $\vec{x} = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 1 \end{pmatrix}$.

Consider this in the form $A\vec{x} = \vec{b}$

where $A = \begin{pmatrix} 10 & -1 & 2 & 0 \\ -1 & 11 & -1 & 3 \\ 2 & -1 & 10 & -1 \\ 0 & 3 & -1 & 8 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 6 \\ 25 \\ -11 \\ 15 \end{pmatrix}$

The idea is to solve $A\vec{x} = \vec{b}$ by rewriting it in the form $\vec{x} = T\vec{x} + \vec{c}$ & then using an iterative scheme of the form $\vec{x}_{(k)} = T\vec{x}_{(k-1)} + \vec{c}$ where $k = 1, 2, 3 \cdots$

Let us re-write $A\vec{x} = b$ as $\vec{x} = T\vec{x} + \vec{c}$ thusly

$$\begin{pmatrix} x_1 = \phantom{-\frac{1}{11}x_1} \quad \frac{1}{10}x_2 - \frac{1}{5}x_3 \phantom{-\frac{3}{11}x_4} + \frac{3}{5} \\ x_2 = \frac{1}{11}x_1 \phantom{+\frac{1}{10}x_2} + \frac{1}{11}x_3 - \frac{3}{11}x_4 + \frac{25}{11} \\ x_3 = -\frac{1}{5}x_1 + \frac{1}{10}x_2 \phantom{+\frac{1}{8}x_3} + \frac{1}{10}x_4 - \frac{11}{10} \\ x_4 = \phantom{-\frac{1}{5}x_1} - \frac{3}{8}x_2 + \frac{1}{8}x_3 \phantom{+\frac{1}{10}x_4} + \frac{15}{8} \end{pmatrix}$$

These unknowns were already computed in the eqns above.

$$\Rightarrow \vec{x} = \begin{pmatrix} 0 & \frac{1}{10} & -\frac{1}{5} & 0 \\ \frac{1}{11} & 0 & \frac{1}{11} & -\frac{3}{11} \\ -\frac{1}{5} & \frac{1}{10} & 0 & \frac{1}{10} \\ 0 & -\frac{3}{8} & \frac{1}{8} & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} 3/5 \\ 25/11 \\ -11/10 \\ 15/8 \end{pmatrix}$$

$$\vec{x} = T\vec{x} + \vec{c}$$

initial guess (say) $\vec{x}_{(0)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ then $\vec{x}_{(1)} = \begin{pmatrix} 0.6 \\ 2.2727 \\ -1.1000 \\ 1.8750 \end{pmatrix}$

Carry forth the computations iteratively

$$\cdots \quad \vec{x}_{(10)} = \begin{pmatrix} 1.0001 \\ 1.9998 \\ -0.9998 \\ 0.9998 \end{pmatrix}$$

Whence $$\frac{\|\vec{x}_{(10)} - \vec{x}_{(9)}\|}{\|\vec{x}_{(10)}\|} < 10^{-3}$$

<u>STOP</u> !

Jacobi iteration (in matrix form)

$$A\vec{x} = b$$

$$\Rightarrow (D+L+U)\vec{x} = b$$

$$\Rightarrow D\vec{x} = -(L+U)\vec{x} + \vec{b}$$

$$\Rightarrow \vec{x} = -D^{-1}(L+U)\vec{x} + \vec{b}$$

Hence the iteration

$$\vec{x}_{(k)} = -D^{-1}(L+U)\vec{x}_{(k-1)} + \vec{b}$$

in the form of iterates we have

$$x_{i(k)} = \frac{\sum_{\substack{j=1 \\ j\neq i}}^{n} \left(-a_{ij}\, x_{j(k-1)}\right) + b_i}{a_{ii}} \; ; \quad i = 1, 2, \ldots, n$$

$a_{ii} \neq 0 \Rightarrow D$ is invertible!

Gauss - Seidel iterative scheme

$$x_{i_{(k)}} = \frac{-\sum_{j=1}^{i-1} a_{ij} x_{j(k)} - \sum_{j=i+1}^{n} a_{ij} x_{j(k-1)} + b_i}{a_{ii}} \quad ; \quad i = 1, 2 \ldots n$$

Why is this a good idea?

Compare w.r.t. the example at the beginning of this lecture!

in matrix form

$$A\vec{x} = \vec{b}$$

$$(D+L)\vec{x} = -U\vec{x} + \vec{b}$$

as iterates $(D+L)\vec{x}_{(k)} = -U\vec{x}_{(k-1)} + \vec{b}$

or $\vec{x}_{(k)} = -(D+L)^{-1}U\vec{x}_{(k-1)} + (D+L)^{-1}\vec{b}$; $k=1,2,\ldots$

Gauss-Seidel iteration

* In Next lecture, we will talk about Convergence of iterative schemes & also SOR scheme!