$$A\mathbf{e} = A(\mathbf{x} - \tilde{\mathbf{x}}) = A\mathbf{x} - A\tilde{\mathbf{x}} = \mathbf{b} - A\tilde{\mathbf{x}} = \mathbf{r}.$$

It follows that for any vector norm $\|\cdot\|$, and the corresponding induced matrix norm, we have

$$\begin{aligned}
\|\mathbf{e}\| &= \|A^{-1}\mathbf{r}\| \\
&\leq \|A^{-1}\|\|\mathbf{r}\| \\
&\leq \|A^{-1}\|\|\mathbf{r}\|\frac{\|\mathbf{b}\|}{\|\mathbf{b}\|} \\
&\leq \|A^{-1}\|\|\mathbf{r}\|\frac{\|A\mathbf{x}\|}{\|\mathbf{b}\|} \\
&\leq \|A\|\|A^{-1}\|\frac{\|\mathbf{r}\|}{\|\mathbf{b}\|}\|\mathbf{x}\|.
\end{aligned}$$

We conclude that the magnitude of the *relative error* in $\tilde{\mathbf{x}}$ is bounded as follows:

$$\frac{\|\mathbf{e}\|}{\|\mathbf{x}\|} \leq \kappa(A)\frac{\|\mathbf{r}\|}{\|\mathbf{b}\|},$$

where

$$\kappa(A) = \|A\|\|A^{-1}\|$$

is the *condition number* of A.

It follows from this bound on the relative error that if $\kappa(A)$ is particularly large, the error in $\tilde{\mathbf{x}}$ can be quite large even if $\mathbf{r}$ is small. Therefore, we say that A is *ill-conditioned.*

** if k(A) ~ 1, then we say A is well conditioned

Since $\tilde{\mathbf{e}}$ is an estimate of the error $\mathbf{e} = \mathbf{x} - \tilde{\mathbf{x}}$ in $\tilde{\mathbf{x}}$, it follows that $\tilde{\mathbf{x}} + \tilde{\mathbf{e}}$ is a more accurate approximation of $\mathbf{x}$ than $\tilde{\mathbf{x}}$ is. This is the basic idea behind *iterative refinement*, also known as *iterative improvement* or *residual correction*. The algorithm is as follows:

$\tilde{\mathbf{x}}^{(0)} = \mathbf{0}$
$\mathbf{r}^{(0)} = \mathbf{b}$
$k = 0$
for $k = 0, 1, 2, \ldots$
    Solve $A\tilde{\mathbf{e}}^{(k)} = \mathbf{r}^{(k)}$
    if $\|\tilde{\mathbf{e}}^{(k)}\|_\infty < TOL$
        break
    end
    $\tilde{\mathbf{x}}^{(k+1)} = \tilde{\mathbf{x}}^{(k)} + \tilde{\mathbf{e}}^{(k)}$
    $\mathbf{r}^{(k+1)} = \mathbf{r}^{(k)} - A\tilde{\mathbf{e}}^{(k)}$
end

Recall: we are solving $Ax = b$

$r := Ae = Ax - A\tilde{x} = b - A\tilde{x}$

$$K(A) \approx \frac{\|\tilde{\mathbf{y}}\|}{\|\tilde{\mathbf{x}}\|} 10^t.$$

—— eq. (7.23)

The linear system given by

$$\begin{bmatrix} 3.3330 & 15920 & -10.333 \\ 2.2220 & 16.710 & 9.6120 \\ 1.5611 & 5.1791 & 1.6852 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 15913 \\ 28.544 \\ 8.4254 \end{bmatrix}$$

has the exact solution $\mathbf{x} = (1, 1, 1)^t$.

Using Gaussian elimination and five-digit rounding arithmetic leads successively to the augmented matrices

$$\begin{bmatrix} 3.3330 & 15920 & -10.333 & 15913 \\ 0 & -10596 & 16.501 & 10580 \\ 0 & -7451.4 & 6.5250 & -7444.9 \end{bmatrix}$$

and

$$\begin{bmatrix} 3.3330 & 15920 & -10.333 & 15913 \\ 0 & -10596 & 16.501 & -10580 \\ 0 & 0 & -5.0790 & -4.7000 \end{bmatrix}.$$

The approximate solution to this system is

$$\tilde{\mathbf{x}} = (1.2001, 0.99991, 0.92538)^t.$$

The residual vector corresponding to $\tilde{\mathbf{x}}$ is computed in double precision to be

$$\mathbf{r} = \mathbf{b} - A\tilde{\mathbf{x}}$$

$$= \begin{bmatrix} 15913 \\ 28.544 \\ 8.4254 \end{bmatrix} - \begin{bmatrix} 3.3330 & 15920 & -10.333 \\ 2.2220 & 16.710 & 9.6120 \\ 1.5611 & 5.1791 & 1.6852 \end{bmatrix} \begin{bmatrix} 1.2001 \\ 0.99991 \\ 0.92538 \end{bmatrix}$$

$$= \begin{bmatrix} 15913 \\ 28.544 \\ 8.4254 \end{bmatrix} - \begin{bmatrix} 15913.00518 \\ 28.26987086 \\ 8.611560367 \end{bmatrix} = \begin{bmatrix} -0.00518 \\ 0.27412914 \\ -0.186160367 \end{bmatrix},$$

so

$$\|\mathbf{r}\|_\infty = 0.27413.$$

The estimate for the condition number given in the preceding discussion is obtained by first solving the system $A\mathbf{y} = \mathbf{r}$ for $\tilde{\mathbf{y}}$:

$$\begin{bmatrix} 3.3330 & 15920 & -10.333 \\ 2.2220 & 16.710 & 9.6120 \\ 1.5611 & 5.1791 & 1.6852 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -0.00518 \\ 0.27413 \\ -0.18616 \end{bmatrix}.$$

This implies that $\tilde{\mathbf{y}} = (-0.20008, 8.9987 \times 10^{-5}, 0.074607)^t$. Using the estimate in Eq. (7.23) gives

$$K(A) \approx \frac{\|\tilde{\mathbf{y}}\|_\infty}{\|\tilde{\mathbf{x}}\|_\infty} 10^5 = \frac{0.20008}{1.2001} 10^5 = 16672. \tag{7.24}$$

To determine the *exact* condition number of A, we first must find A^{-1}. Using five-digit rounding arithmetic for the calculations gives the approximation:

$$A^{-1} \approx \begin{bmatrix} -1.1701 \times 10^{-4} & -1.4983 \times 10^{-1} & 8.5416 \times 10^{-1} \\ 6.2782 \times 10^{-5} & 1.2124 \times 10^{-4} & -3.0662 \times 10^{-4} \\ -8.6631 \times 10^{-5} & 1.3846 \times 10^{-1} & -1.9689 \times 10^{-1} \end{bmatrix}.$$

$$\|A^{-1}\|_\infty = 1.0041 \text{ and } \|A\|_\infty = 15934.$$

As a consequence, the ill-conditioned matrix A has

$$K(A) = (1.0041)(15934) = 15999.$$

The estimate in (7.24) is quite close to $K(A)$ and requires considerably less computational effort.

Since the actual solution $\mathbf{x} = (1, 1, 1)^t$ is known for this system, we can calculate both

$$\|\mathbf{x} - \tilde{\mathbf{x}}\|_\infty = 0.2001 \quad \text{and} \quad \frac{\|\mathbf{x} - \tilde{\mathbf{x}}\|_\infty}{\|\mathbf{x}\|_\infty} = \frac{0.2001}{1} = 0.2001.$$

The error bounds

$$\|\mathbf{x} - \tilde{\mathbf{x}}\|_\infty \le K(A)\frac{\|\mathbf{r}\|_\infty}{\|A\|_\infty} = \frac{(15999)(0.27413)}{15934} = 0.27525$$

and

$$\frac{\|\mathbf{x} - \tilde{\mathbf{x}}\|_\infty}{\|\mathbf{x}\|_\infty} \le K(A)\frac{\|\mathbf{r}\|_\infty}{\|\mathbf{b}\|_\infty} = \frac{(15999)(0.27413)}{15913} = 0.27561. \quad \square$$