Iterative Techniques in Matrix Algebra

Jacobi & Gauss-Seidel Iterative Techniques II

Numerical Analysis (9th Edition) R L Burden & J D Faires

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Gauss-Seidel Method	Gauss-Seidel Algorithm	Convergence Results	Interpretation
Outline			



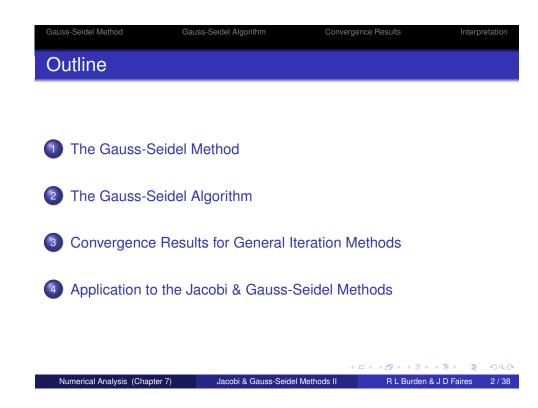
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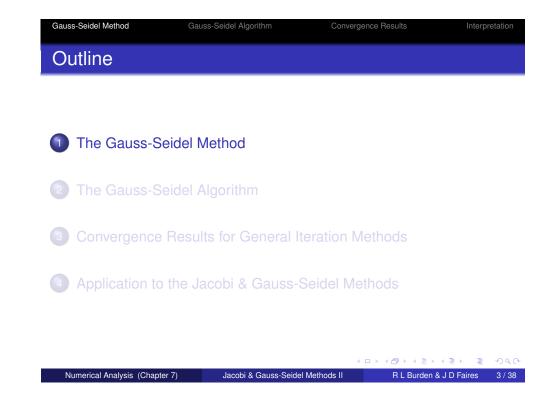
Gauss-Seidel Method	Gauss-Seidel Algorithm	Convergence Results	Interpretation
Outline			
1 The Gauss-S	eidel Method		
2 The Gauss-S	eidel Algorithm		

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Gauss-Seidel Method	Gauss-Seidel Algorithm	Convergence Results	Interpretation
Outline			
1 The Gauss-S	eidel Method		
2 The Gauss-S	eidel Algorithm		
3 Convergence	Results for General It	eration Methods	

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The Gauss-Seidel Method

Gauss-Seidel Algorithm

Looking at the Jacobi Method

Gauss-Seidel Method

• A possible improvement to the Jacobi Algorithm can be seen by re-considering

$$x_{i}^{(k)} = \frac{1}{a_{ii}} \left[\sum_{\substack{j=1\\j\neq i}}^{n} \left(-a_{ij} x_{j}^{(k-1)} \right) + b_{i} \right], \quad \text{for } i = 1, 2, \dots, n$$

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The Gauss-Seidel Method

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Gauss-Seidel Method

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$$x_i^{(k)} = \frac{1}{a_{ij}} \left[\sum_{\substack{j=1 \ j \neq i}}^n \left(-a_{ij} x_j^{(k-1)} \right) + b_i \right], \quad \text{for } i = 1, 2, \dots, n$$

• The components of $\mathbf{x}^{(k-1)}$ are used to compute all the components $x_i^{(k)}$ of $\mathbf{x}^{(k)}$.

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The Gauss-Seidel Method

Looking at the Jacobi Method

Gauss-Seidel Method

• A possible improvement to the Jacobi Algorithm can be seen by re-considering

$$x_i^{(k)} = \frac{1}{a_{ij}} \left[\sum_{\substack{j=1 \ j \neq i}}^n \left(-a_{ij} x_j^{(k-1)} \right) + b_i \right], \quad \text{for } i = 1, 2, \dots, n$$

- The components of x^(k-1) are used to compute all the components x^(k)_i of x^(k).
- But, for *i* > 1, the components x₁^(k),..., x_{i-1}^(k) of **x**^(k) have already been computed and are expected to be better approximations to the actual solutions x₁,..., x_{i-1} than are x₁^(k-1),..., x_{i-1}^(k-1).

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Gauss-Seidel MethodGauss-Seidel AlgorithmConvergence ResultsInterpretationThe Gauss-Seidel MethodInstead of using $x_i^{(k)} = \frac{1}{a_{ii}} \left[\sum_{\substack{j=1\\j\neq i}}^n \left(-a_{ij} x_j^{(k-1)} \right) + b_i \right], \quad \text{for } i = 1, 2, \dots, n$ it seems reasonable, then, to compute $x_i^{(k)}$ using these most recently calculated values.

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Gauss-Seidel Method Gauss-Seidel Algorithm Convergence Results The Gauss-Seidel Method Instead of using

$$x_i^{(k)} = rac{1}{a_{ij}} \left[\sum_{\substack{j=1\ j
eq i}}^n \left(-a_{ij} x_j^{(k-1)}
ight) + b_i
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it seems reasonable, then, to compute $x_i^{(k)}$ using these most recently calculated values.

The Gauss-Seidel Iterative Technique									
$x_i^{(k)} = \frac{1}{a_{ii}}$	$\left[-\sum_{j=1}^{i-1} (a_{ij}x_j^{(k)}) - \sum_{j=i+1}^n (a_{ij}x_j^{(k)})\right]$	$\binom{(k-1)}{j} + b_i$							
for each $i = 1, 2,,$	n.								
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Gauss-Seidel MethodGauss-Seidel AlgorithmConvergence ResultsInterpretationThe Gauss-Seidel MethodExampleUse the Gauss-Seidel iterative technique to find approximate solutions to $10x_1 - x_2 + 2x_3 = 6$ $-x_1 + 11x_2 - x_3 + 3x_4 = 25$ $2x_1 - x_2 + 10x_3 - x_4 = -11'$ $3x_2 - x_3 + 8x_4 = 15$

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Gauss-Seidel Method	Gauss-Seidel Algorithm	Convergence Results	Interpretation
The Gauss-	Seidel Method		
Example			
Use the Gauss- to starting with x =		$x_3 = 6$ $x_3 + 3x_4 = 25$	e solutions

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Gauss-Seidel MethodGauss-Seidel AlgorithmConvergence ResultsInterpretationThe Gauss-Seidel MethodExampleUse the Gauss-Seidel iterative technique to find approximate solutions to $10x_1 - x_2 + 2x_3 = 6$
 $-x_1 + 11x_2 - x_3 + 3x_4 = 25$
 $2x_1 - x_2 + 10x_3 - x_4 = -11'$
 $3x_2 - x_3 + 8x_4 = 15$ starting with $\mathbf{x} = (0, 0, 0, 0)^t$ and iterating until $\frac{\|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\|_{\infty}}{\|\mathbf{x}^{(k)}\|_{\infty}} < 10^{-3}$

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Gauss-Seidel MethodGauss-Seidel AlgorithmConvergence ResultsInterpretationThe Gauss-Seidel MethodExampleUse the Gauss-Seidel iterative technique to find approximate solutions to $10x_1 - x_2 + 2x_3 = 6$ $-x_1 + 11x_2 - x_3 + 3x_4 = 25$ $2x_1 - x_2 + 10x_3 - x_4 = -11'$ $3x_2 - x_3 + 8x_4 = 15$ starting with $\mathbf{x} = (0, 0, 0, 0)^t$ and iterating until $\frac{\|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\|_{\infty}}{\|\mathbf{x}^{(k)}\|_{\infty}} < 10^{-3}$

Note: The solution $\mathbf{x} = (1, 2, -1, 1)^t$ was approximated by Jacobi's method in an earlier example.

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Solution (1/3)

For the Gauss-Seidel method we write the system, for each k = 1, 2, ... as

$x_{1}^{(k)} =$	$\frac{1}{10}x_2^{(k-1)}$ -	$\frac{1}{5}x_3^{(k-1)}$	$+\frac{3}{5}$
$x_2^{(k)} = \frac{1}{11} x_1^{(k)}$	+ -	$\frac{1}{11}x_3^{(k-1)} - \frac{3}{11}x_4^{(k-1)}$	$(1) + \frac{25}{11}$
$x_3^{(k)} = -\frac{1}{5}x_1^{(k)}$	$+\frac{1}{10}x_{2}^{(k)}$	$+ \frac{1}{10} x_4^{(k-1)}$	$(1) - \frac{11}{10}$
$x_{4}^{(k)} =$	$-\frac{3}{8}x_2^{(k)}$ +	$\frac{1}{8}x_3^{(k)}$	$+\frac{15}{8}$

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Gauss-Seidel Method Gauss-Seidel Algorithm Convergence Results Interpretation The Gauss-Seidel Method Interpretation Interpretation

Solution (2/3)

When $\mathbf{x}^{(0)} = (0, 0, 0, 0)^t$, we have $\mathbf{x}^{(1)} = (0.6000, 2.3272, -0.9873, 0.8789)^t$.

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Gauss-Seidel Method The Gauss-Seidel Method

Solution (2/3)

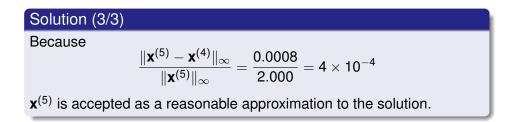
When $\mathbf{x}^{(0)} = (0, 0, 0, 0)^t$, we have $\mathbf{x}^{(1)} = (0.6000, 2.3272, -0.9873, 0.8789)^t$. Subsequent iterations give the values in the following table:

k	0	1	2	3	4	5
$x_{1}^{(k)}$	0.0000	0.6000	1.030	1.0065	1.0009	1.0001
$x_{2}^{(k)}$				2.0036	2.0003	2.0000
$x_{3}^{(k)}$	0.0000				-1.0003	-1.0000
$x_{4}^{(k)}$	0.0000	0.8789	0.984	0.9983	0.9999	1.0000

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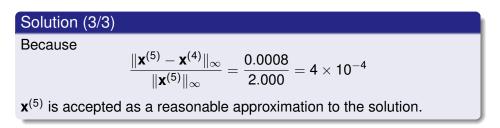
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Gauss-Seidel Method Gauss-Seidel Algorithm Convergence Results Interpretation The Gauss-Seidel Method Interpretation Interpretation



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Gauss-Seidel Method Gauss-Seidel Algorithm Convergence Results Interpretation The Gauss-Seidel Method Interpretation Interpretation



Note that, in an earlier example, Jacobi's method required twice as many iterations for the same accuracy.

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Gauss-Seidel Method Gauss-Seidel Algorithm Convergence Results Interpretation The Gauss-Seidel Method: Matrix Form

Re-Writing the Equations
To write the Gauss-Seidel method in matrix form,

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Gauss-Seidel Method Gauss-Seidel Algorithm Convergence Results Interpretation The Gauss-Seidel Method: Matrix Form

Re-Writing the Equations

To write the Gauss-Seidel method in matrix form, multiply both sides of

$$x_i^{(k)} = rac{1}{a_{ii}} \left[-\sum_{j=1}^{i-1} (a_{ij} x_j^{(k)}) - \sum_{j=i+1}^n (a_{ij} x_j^{(k-1)}) + b_i
ight]$$

by *a_{ii}* and collect all *k*th iterate terms,

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Gauss-Seidel Method Gauss-Seidel Algorithm Convergence Results Interpretation The Gauss-Seidel Method: Matrix Form

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ight]$$

by a_{ii} and collect all kth iterate terms, to give

$$a_{i1}x_1^{(k)} + a_{i2}x_2^{(k)} + \dots + a_{ii}x_i^{(k)} = -a_{i,i+1}x_{i+1}^{(k-1)} - \dots - a_{in}x_n^{(k-1)} + b_i$$

for each $i = 1, 2, \dots, n$.

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Gauss-Seidel MethodGauss-Seidel AlgorithmConvergence ResultsInterpretationThe Gauss-Seidel Method: Matrix FormRe-Writing the Equations (Cont'd)Writing all *n* equations gives $a_{11}x_1^{(k)}$ $= -a_{12}x_2^{(k-1)} - a_{13}x_3^{(k-1)} - \cdots - a_{1n}x_n^{(k-1)} + b_1$ $a_{21}x_1^{(k)}$ $= -a_{23}x_3^{(k-1)} - \cdots - a_{2n}x_n^{(k-1)} + b_2$

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$a_{n1}x_1^{(k)}$	+	$a_{n2}x_2^{(k)} + \dots + a_{nn}x_n^{(k)}$	=		b_n
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Gauss-Seidel MethodGauss-Seidel AlgorithmConvergence ResultsInterpretationThe Gauss-Seidel Method: Matrix FormRe-Writing the Equations (Cont'd)Writing all *n* equations gives $a_{11}x_1^{(k)}$ $= -a_{12}x_2^{(k-1)} - a_{13}x_3^{(k-1)} - \cdots - a_{1n}x_n^{(k-1)} + b_1$ $a_{21}x_1^{(k)}$ $= -a_{22}x_2^{(k-1)} - a_{13}x_3^{(k-1)} - \cdots - a_{2n}x_n^{(k-1)} + b_2$ \vdots $= -a_{12}x_1^{(k-1)} - a_{13}x_3^{(k-1)} - \cdots - a_{2n}x_n^{(k-1)} + b_2$ \vdots $= -a_{12}x_1^{(k-1)} - a_{13}x_3^{(k-1)} - \cdots - a_{2n}x_n^{(k-1)} + b_2$ \vdots $= -a_{12}x_1^{(k-1)} - a_{13}x_3^{(k-1)} - \cdots - a_{2n}x_n^{(k-1)} + b_2$ \vdots $= -a_{12}x_1^{(k-1)} - a_{13}x_3^{(k-1)} - \cdots - a_{2n}x_n^{(k-1)} + b_2$ \vdots $= -a_{12}x_1^{(k-1)} - a_{13}x_3^{(k-1)} - \cdots - a_{2n}x_n^{(k-1)} + b_2$ \vdots $= -a_{12}x_1^{(k-1)} - a_{13}x_3^{(k-1)} - \cdots - a_{2n}x_n^{(k-1)} + b_2$ \vdots $= -a_{12}x_1^{(k-1)} - a_{13}x_3^{(k-1)} - \cdots - a_{2n}x_n^{(k-1)} + b_2$ \vdots $= -a_{12}x_1^{(k)} + a_{12}x_2^{(k)} + \cdots + a_{nn}x_n^{(k)} = b_n$

With the definitions of D, L, and U given previously, we have the Gauss-Seidel method represented by

$$(D-L)\mathbf{x}^{(k)} = U\mathbf{x}^{(k-1)} + \mathbf{k}$$

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$$(D-L)\mathbf{x}^{(k)} = U\mathbf{x}^{(k-1)} + \mathbf{b}$$

Re-Writing the Equations (Cont'd)

Solving for $\mathbf{x}^{(k)}$ finally gives

$$\mathbf{x}^{(k)} = (D-L)^{-1} U \mathbf{x}^{(k-1)} + (D-L)^{-1} \mathbf{b}$$
, for each $k = 1, 2, ...$

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Gauss-Seidel Method Gauss-Seidel Algorithm Convergence Results Interpretation The Gauss-Seidel Method: Matrix Form

$$(D-L){\bf x}^{(k)} = U{\bf x}^{(k-1)} + {\bf b}$$

Re-Writing the Equations (Cont'd)

Solving for $\mathbf{x}^{(k)}$ finally gives

$$\mathbf{x}^{(k)} = (D-L)^{-1} U \mathbf{x}^{(k-1)} + (D-L)^{-1} \mathbf{b}, \text{ for each } k = 1, 2, ...$$

Letting $T_g = (D - L)^{-1}U$ and $\mathbf{c}_g = (D - L)^{-1}\mathbf{b}$, gives the Gauss-Seidel technique the form

$$\mathbf{x}^{(k)} = T_g \mathbf{x}^{(k-1)} + \mathbf{c}_g$$

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$$(D-L)\mathbf{x}^{(k)} = U\mathbf{x}^{(k-1)} + \mathbf{b}$$

Re-Writing the Equations (Cont'd)

Solving for $\mathbf{x}^{(k)}$ finally gives

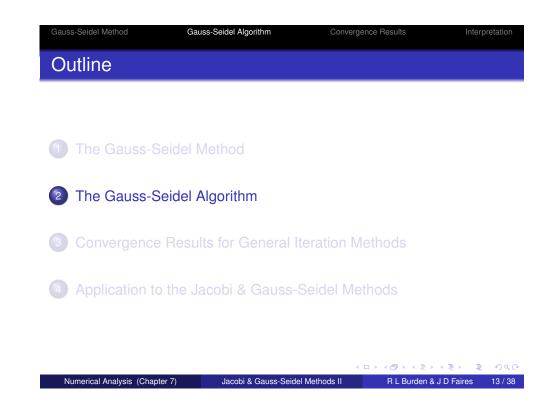
$$\mathbf{x}^{(k)} = (D-L)^{-1} U \mathbf{x}^{(k-1)} + (D-L)^{-1} \mathbf{b}, \text{ for each } k = 1, 2, ...$$

Letting $T_g = (D - L)^{-1}U$ and $\mathbf{c}_g = (D - L)^{-1}\mathbf{b}$, gives the Gauss-Seidel technique the form

$$\mathbf{x}^{(k)} = T_g \mathbf{x}^{(k-1)} + \mathbf{c}_g$$

For the lower-triangular matrix D - L to be nonsingular, it is necessary and sufficient that $a_{ii} \neq 0$, for each i = 1, 2, ..., n.

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Gauss-Seidel Method	Gauss-Seidel Algorithm	Convergence Results	Interpretation
Gauss-Seide	I Iterative Algorith	nm (1/2)	

To solve $A\mathbf{x} = \mathbf{b}$ given an initial approximation $\mathbf{x}^{(0)}$:

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Gauss-Seidel Method	Gauss-Seidel Algorithm	Convergence Results	Interpretation
Gauss-Sei	idel Iterative Algorit	hm (1/2)	
To solve A x =	= b given an initial approx	kimation $\mathbf{x}^{(0)}$:	
INPUT	the number of equation the entries a_{ij} , $1 \le i, j \le$ the entries b_i , $1 \le i \le r$ the entries XO_i , $1 \le i \le$ tolerance TOL ; maximum number of ite	$\leq n$ of the matrix A ; n of b ; $\leq n$ of XO = x ⁽⁰⁾ ;	

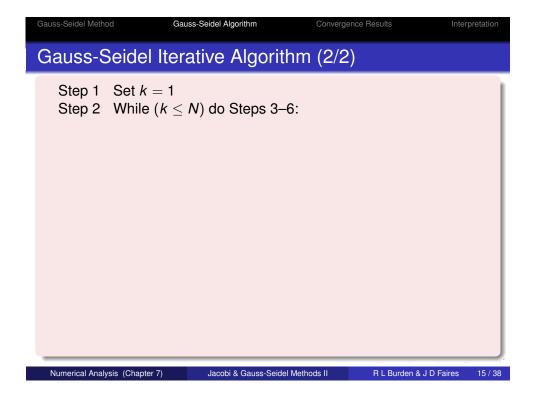
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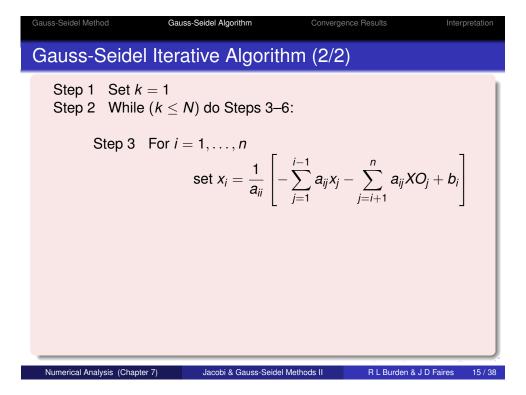
Gauss-Seidel Method	Gauss-Seidel Algorithm	Convergence Results	Interpretation
Gauss-Seide	I Iterative Algorit	nm (1/2)	

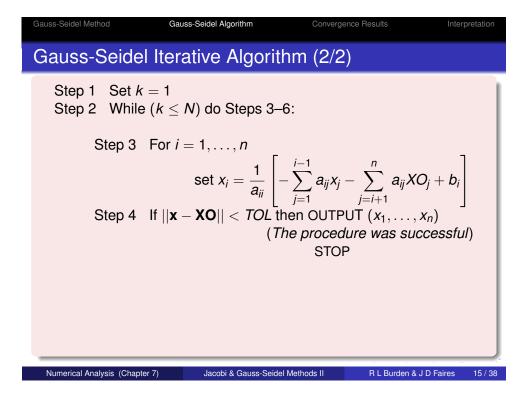
To solve $A\mathbf{x} = \mathbf{b}$ given an initial approximation $\mathbf{x}^{(0)}$:

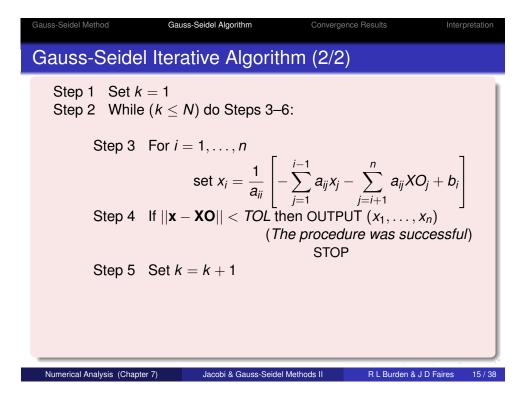
- INPUT the number of equations and unknowns n; the entries a_{ij} , $1 \le i, j \le n$ of the matrix A; the entries b_i , $1 \le i \le n$ of **b**; the entries XO_i , $1 \le i \le n$ of **XO** = **x**⁽⁰⁾; tolerance *TOL*; maximum number of iterations N. OUTPUT the approximate solution x_1, \ldots, x_n or a message
 - that the number of iterations was exceeded.

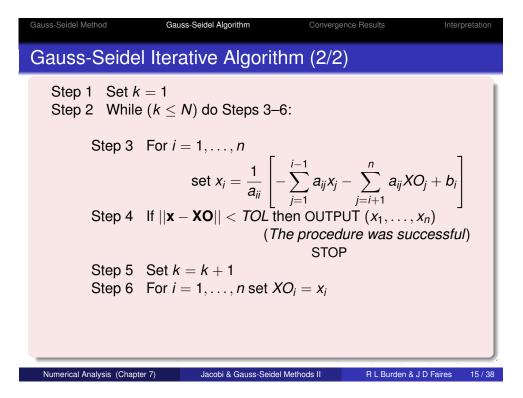
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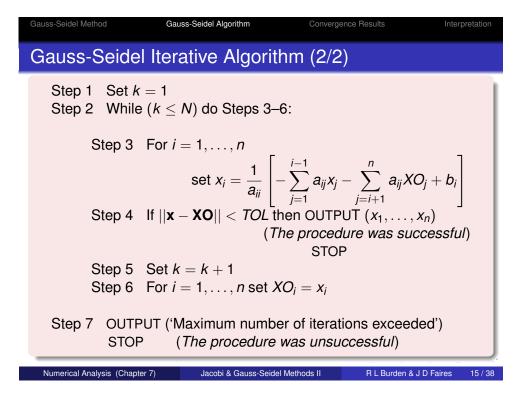


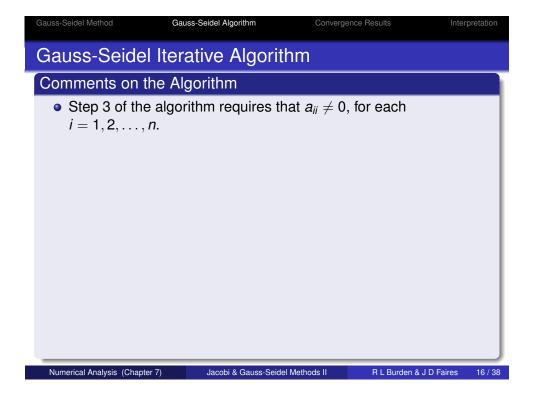


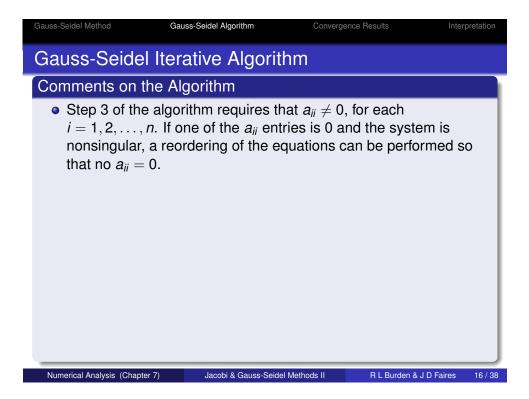


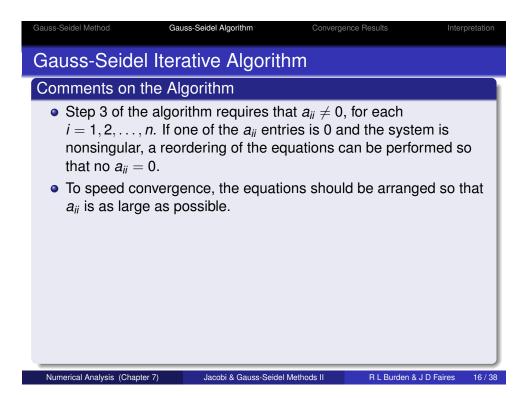


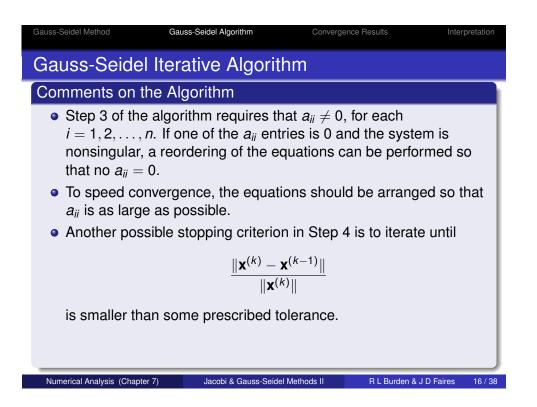


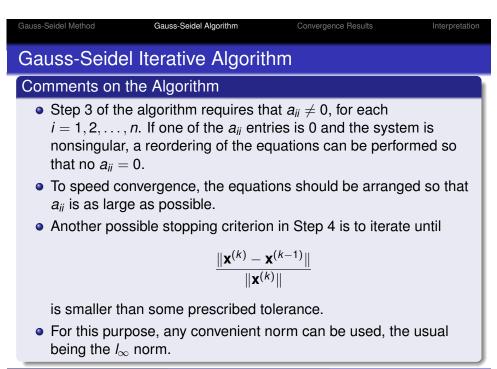




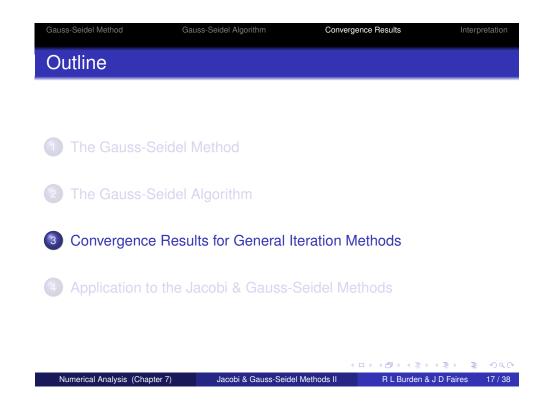








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Introduction

• To study the convergence of general iteration techniques, we need to analyze the formula

$$\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c}$$
, for each $k = 1, 2, ...$

where $\mathbf{x}^{(0)}$ is arbitrary.

• The following lemma and the earlier • Theorem on convergent matrices provide the key for this study.

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Lemma

If the spectral radius satisfies $\rho(T) < 1$, then $(I - T)^{-1}$ exists, and

$$(I - T)^{-1} = I + T + T^2 + \dots = \sum_{j=0}^{\infty} T^j$$

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$$(I - T)^{-1} = I + T + T^2 + \dots = \sum_{j=0}^{\infty} T^j$$

Proof (1/2)

• Because $T\mathbf{x} = \lambda \mathbf{x}$ is true precisely when $(I - T)\mathbf{x} = (1 - \lambda)\mathbf{x}$, we have λ as an eigenvalue of T precisely when $1 - \lambda$ is an eigenvalue of I - T.

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$$(I - T)^{-1} = I + T + T^2 + \dots = \sum_{j=0}^{\infty} T^j$$

Proof (1/2)

- Because Tx = λx is true precisely when (I − T)x = (1 − λ)x, we have λ as an eigenvalue of T precisely when 1 − λ is an eigenvalue of I − T.
- But |λ| ≤ ρ(T) < 1, so λ = 1 is not an eigenvalue of T, and 0 cannot be an eigenvalue of I − T.

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- But |λ| ≤ ρ(T) < 1, so λ = 1 is not an eigenvalue of T, and 0 cannot be an eigenvalue of I − T.
- Hence, $(I T)^{-1}$ exists.



Proof (2/2)	
Let	$2 + \pi^2 + \pi^2$
	$S_m = I + T + T^2 + \cdots + T^m$

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Proof (2/2)
Let
$S_m = I + T + T^2 + \cdots + T^m$
Then
$(I-T)S_m = (1+T+T^2+\cdots+T^m) - (T+T^2+\cdots+T^{m+1}) = I-T^{m+1}$

|--|

Proof (2/2)

Let

 $S_m = I + T + T^2 + \cdots + T^m$

Then

$$(I-T)S_m = (1+T+T^2+\cdots+T^m) - (T+T^2+\cdots+T^{m+1}) = I-T^{m+1}$$

and, since T is convergent, the \frown merem on convergent matrices implies that

$$\lim_{m\to\infty}(I-T)S_m = \lim_{m\to\infty}(I-T^{m+1}) = I$$

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Proof (2/2)

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$$S_m = I + T + T^2 + \cdots + T^m$$

Then

$$(I-T)S_m = (1+T+T^2+\cdots+T^m) - (T+T^2+\cdots+T^{m+1}) = I-T^{m+1}$$

and, since T is convergent, the \bullet Theorem on convergent matrices implies that

$$\lim_{m \to \infty} (I - T)S_m = \lim_{m \to \infty} (I - T^{m+1}) = I$$

Thus, $(I - T)^{-1} = \lim_{m \to \infty} S_m = I + T + T^2 + \dots = \sum_{j=0}^{\infty} T^j$

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TheoremFor any $\mathbf{x}^{(0)} \in \mathbb{R}^n$, the sequence $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ defined by $\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c}$, for each $k \ge 1$ converges to the unique solution of $\mathbf{x} = T\mathbf{x} + \mathbf{c}$ if and only if $\rho(T) < 1$.

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Proof (1/5)			
First assume that $\rho(T)$	< 1.		
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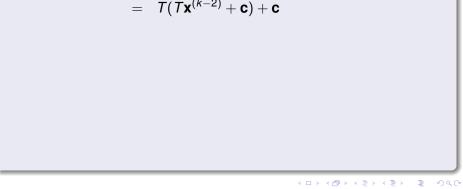
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Proof (1/5)

First assume that $\rho(T) < 1$. Then,

$$\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c}$$

= $T(T\mathbf{x}^{(k-2)} + \mathbf{c}) + \mathbf{c}$





Proof (1/5)

First assume that $\rho(T) < 1$. Then,

$$\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c}$$

= $T(T\mathbf{x}^{(k-2)} + \mathbf{c}) + \mathbf{c}$
- $T^2\mathbf{x}^{(k-2)} + (T + t)\mathbf{c}$

$$= T^2 \mathbf{x}^{(k-2)} + (T+I)\mathbf{c}$$

Gauss-Seidel Method Convergence Results Convergence Results for General Iteration Methods

Proof (1/5)

First assume that $\rho(T) < 1$. Then, $\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c}$ $= T(T\mathbf{x}^{(k-2)} + \mathbf{c}) + \mathbf{c}$ $= T^2 \mathbf{x}^{(k-2)} + (T+I)\mathbf{c}$ ÷ $= T^{k} \mathbf{x}^{(0)} + (T^{k-1} + \dots + T + I) \mathbf{c}$ ▲□▶▲圖▶▲≣▶▲≣▶ = ● ● ●

R L Burden & J D Faires 22 / 38 Numerical Analysis (Chapter 7) Jacobi & Gauss-Seidel Methods II

Proof (1/5)

First assume that $\rho(T) < 1$. Then, $\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c}$ $= T(T\mathbf{x}^{(k-2)} + \mathbf{c}) + \mathbf{c}$ $= T^2\mathbf{x}^{(k-2)} + (T+I)\mathbf{c}$ \vdots $= T^k\mathbf{x}^{(0)} + (T^{k-1} + \dots + T+I)\mathbf{c}$ Because $\rho(T) < 1$, the Theorem on convergent matrices implies that T is convergent, and $\lim_{k \to \infty} T^k \mathbf{x}^{(0)} = \mathbf{0}$ Mumerical Analysis (Chapter 7) Acceleration of the convergent of the table as a convergent of table as a co

Proof (2/5)

The previous lemma implies that

$$\lim_{k\to\infty} \mathbf{x}^{(k)} = \lim_{k\to\infty} T^k \mathbf{x}^{(0)} + \left(\sum_{j=0}^{\infty} T^j\right) \mathbf{c}$$

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$$= \mathbf{0} + (I - T)^{-1} \mathbf{c}$$
$$= (I - T)^{-1} \mathbf{c}$$
we the sequence $\{\mathbf{x}^{(k)}\}$ converges to the vector $\mathbf{x} = (I - T)^{-1}$

Hence, the sequence $\{\mathbf{x}^{(k)}\}$ converges to the vector $\mathbf{x} \equiv (I - T)^{-1}\mathbf{c}$ and $\mathbf{x} = T\mathbf{x} + \mathbf{c}$.

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Proof (3/5) • To prove the converse, we will show that for any $\mathbf{z} \in \mathbb{R}^n$, we have $\lim_{k\to\infty} T^k \mathbf{z} = \mathbf{0}$.

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- To prove the converse, we will show that for any $z \in \mathbb{R}^n$, we have $\lim_{k\to\infty} T^k z = 0$.
- Again, by the theorem on convergent matrices, this is equivalent to ρ(T) < 1.

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- Let \mathbf{z} be an arbitrary vector, and \mathbf{x} be the unique solution to $\mathbf{x} = T\mathbf{x} + \mathbf{c}$.

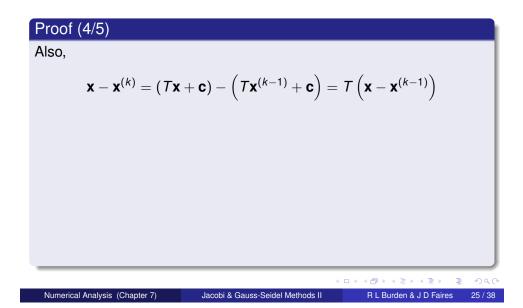
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- Define $\mathbf{x}^{(0)} = \mathbf{x} \mathbf{z}$, and, for $k \ge 1$, $\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c}$.

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- Define $\mathbf{x}^{(0)} = \mathbf{x} \mathbf{z}$, and, for $k \ge 1$, $\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c}$.
- Then $\{\mathbf{x}^{(k)}\}$ converges to **x**.

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Proof (4/5) Also, $\mathbf{x} - \mathbf{x}^{(k)} = (T\mathbf{x} + \mathbf{c}) - (T\mathbf{x}^{(k-1)} + \mathbf{c}) = T(\mathbf{x} - \mathbf{x}^{(k-1)})$ So $\mathbf{x} - \mathbf{x}^{(k)} = T(\mathbf{x} - \mathbf{x}^{(k-1)})$ $e^{-\mathbf{c} \cdot \mathbf{c}} \cdot \mathbf{c} \cdot \mathbf{c} \in \mathbf{c} \in \mathbf{c} \in \mathbf{c}$

Proof (4/5)

2

Also,

$$\mathbf{x} - \mathbf{x}^{(k)} = (T\mathbf{x} + \mathbf{c}) - (T\mathbf{x}^{(k-1)} + \mathbf{c}) = T(\mathbf{x} - \mathbf{x}^{(k-1)})$$

SO

$$\mathbf{x} - \mathbf{x}^{(k)} = T\left(\mathbf{x} - \mathbf{x}^{(k-1)}\right)$$
$$= T^2\left(\mathbf{x} - \mathbf{x}^{(k-2)}\right)$$

|--|

Proof (4/5)

2

Also,

$$\mathbf{x} - \mathbf{x}^{(k)} = (T\mathbf{x} + \mathbf{c}) - \left(T\mathbf{x}^{(k-1)} + \mathbf{c}\right) = T\left(\mathbf{x} - \mathbf{x}^{(k-1)}\right)$$

so

$$\mathbf{x} - \mathbf{x}^{(k)} = T\left(\mathbf{x} - \mathbf{x}^{(k-1)}\right)$$
$$= T^{2}\left(\mathbf{x} - \mathbf{x}^{(k-2)}\right)$$
$$= \vdots$$

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Proof (4/5)

Also,

$$\mathbf{x} - \mathbf{x}^{(k)} = (T\mathbf{x} + \mathbf{c}) - (T\mathbf{x}^{(k-1)} + \mathbf{c}) = T(\mathbf{x} - \mathbf{x}^{(k-1)})$$

so

$$\mathbf{x} - \mathbf{x}^{(k)} = T\left(\mathbf{x} - \mathbf{x}^{(k-1)}\right)$$
$$= T^{2}\left(\mathbf{x} - \mathbf{x}^{(k-2)}\right)$$
$$= \vdots$$
$$= T^{k}\left(\mathbf{x} - \mathbf{x}^{(0)}\right)$$

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Proof (4/5)

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Also,

$$\mathbf{x} - \mathbf{x}^{(k)} = (T\mathbf{x} + \mathbf{c}) - (T\mathbf{x}^{(k-1)} + \mathbf{c}) = T(\mathbf{x} - \mathbf{x}^{(k-1)})$$

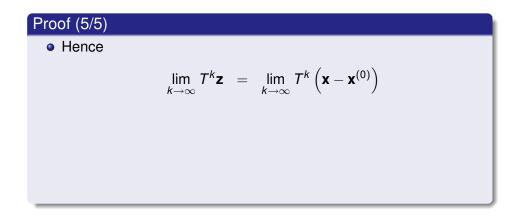
SO

$$\mathbf{x} - \mathbf{x}^{(k)} = T\left(\mathbf{x} - \mathbf{x}^{(k-1)}\right)$$
$$= T^{2}\left(\mathbf{x} - \mathbf{x}^{(k-2)}\right)$$
$$= \vdots$$
$$= T^{k}\left(\mathbf{x} - \mathbf{x}^{(0)}\right)$$
$$= T^{k}\mathbf{z}$$

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Gauss-Seidel Method	Gauss-Seidel Algorithm	Convergence Results	Interpretation
Convergence F	Results for Gener	al Iteration Me	thods



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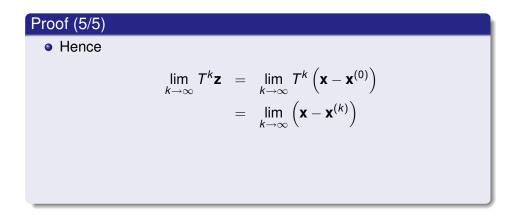
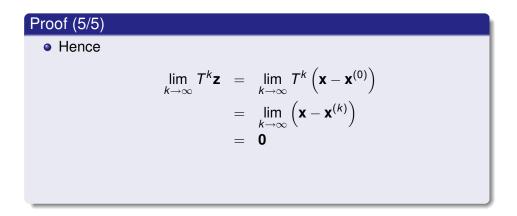


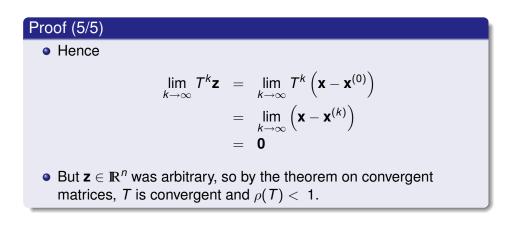
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Corollary

||T|| < 1 for any natural matrix norm and **c** is a given vector, then the sequence $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ defined by

$$\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c}$$

converges, for any $\mathbf{x}^{(0)} \in \mathbb{R}^n$, to a vector $\mathbf{x} \in \mathbb{R}^n$, with $\mathbf{x} = T\mathbf{x} + \mathbf{c}$, and the following error bounds hold:

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(i)
$$\|\mathbf{x} - \mathbf{x}^{(k)}\| \le \|T\|^k \|\mathbf{x}^{(0)} - \mathbf{x}\|$$

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Numerical Analysis (Chapter 7)	Jacobi & Gauss-Seidel Methods II	F	R L Burden &	J D Faires	5	27 / 38

Corollary

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(i)
$$\|\mathbf{x} - \mathbf{x}^{(k)}\| \le \|\mathcal{T}\|^{k} \|\mathbf{x}^{(0)} - \mathbf{x}\|$$

(ii) $\|\mathbf{x} - \mathbf{x}^{(k)}\| \le \frac{\|\mathcal{T}\|^{k}}{1 - \|\mathcal{T}\|} \|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}$

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Numerical Analysis (Chapter 7)	Jacobi & Gauss-Seidel Methods II		R L Burden & J D Faires	27 / 38

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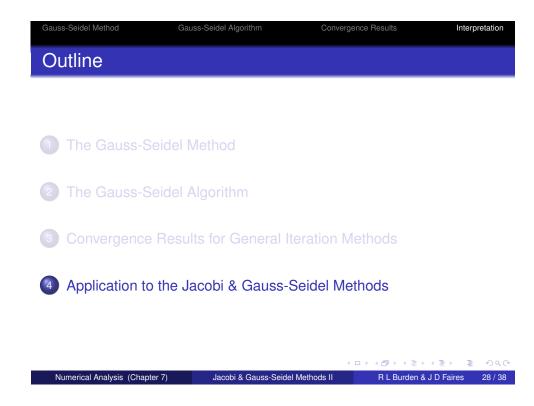
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The proof of the following corollary is similar to that for the • Corollary to the Fixed-Point Theorem for a single nonlinear equation.

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Using the Matrix Formulations

We have seen that the Jacobi and Gauss-Seidel iterative techniques can be written

$$\mathbf{x}^{(k)} = T_j \mathbf{x}^{(k-1)} + \mathbf{c}_j$$
 and
 $\mathbf{x}^{(k)} = T_a \mathbf{x}^{(k-1)} + \mathbf{c}_a$

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Using the Matrix Formulations

We have seen that the Jacobi and Gauss-Seidel iterative techniques can be written

$$\mathbf{x}^{(k)} = T_j \mathbf{x}^{(k-1)} + \mathbf{c}_j \text{ and}$$

$$\mathbf{x}^{(k)} = T_g \mathbf{x}^{(k-1)} + \mathbf{c}_g$$

using the matrices

$$T_j = D^{-1}(L+U)$$
 and $T_g = (D-L)^{-1}U$

respectively.

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Using the Matrix Formulations

We have seen that the Jacobi and Gauss-Seidel iterative techniques can be written

$$\mathbf{x}^{(k)} = T_j \mathbf{x}^{(k-1)} + \mathbf{c}_j$$
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using the matrices

$$T_j = D^{-1}(L+U)$$
 and $T_g = (D-L)^{-1}U$

respectively. If $\rho(T_j)$ or $\rho(T_g)$ is less than 1, then the corresponding sequence $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ will converge to the solution \mathbf{x} of $A\mathbf{x} = \mathbf{b}$.

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Example

For example, the Jacobi method has

$$\mathbf{x}^{(k)} = D^{-1}(L+U)\mathbf{x}^{(k-1)} + D^{-1}\mathbf{b},$$

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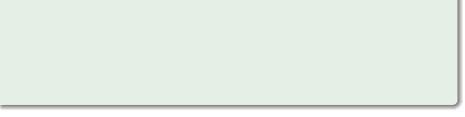


Example

For example, the Jacobi method has

$$\mathbf{x}^{(k)} = D^{-1}(L+U)\mathbf{x}^{(k-1)} + D^{-1}\mathbf{b},$$

and, if $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ converges to \mathbf{x} ,





Example

For example, the Jacobi method has

$$\mathbf{x}^{(k)} = D^{-1}(L+U)\mathbf{x}^{(k-1)} + D^{-1}\mathbf{b},$$

and, if $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ converges to \mathbf{x} , then

$$x = D^{-1}(L+U)x + D^{-1}k$$

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Example

For example, the Jacobi method has

$$\mathbf{x}^{(k)} = D^{-1}(L+U)\mathbf{x}^{(k-1)} + D^{-1}\mathbf{b}$$

and, if $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ converges to \mathbf{x} , then

$$x = D^{-1}(L+U)x + D^{-1}k$$

This implies that

$$D\mathbf{x} = (L+U)\mathbf{x} + \mathbf{b}$$
 and $(D-L-U)\mathbf{x} = \mathbf{b}$

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Example

For example, the Jacobi method has

$$\mathbf{x}^{(k)} = D^{-1}(L+U)\mathbf{x}^{(k-1)} + D^{-1}\mathbf{b},$$

and, if $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ converges to \mathbf{x} , then

$$x = D^{-1}(L+U)x + D^{-1}b$$

This implies that

$$D\mathbf{x} = (L+U)\mathbf{x} + \mathbf{b}$$
 and $(D-L-U)\mathbf{x} = \mathbf{b}$

Since D - L - U = A, the solution **x** satisfies A**x** = **b**.

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	Numerical Analysis (Chapter 7)	Jacobi & Gauss-Seidel Methods II	R L Burden & J D Faires	30 / 38

The following are easily verified sufficiency conditions for convergence of the Jacobi and Gauss-Seidel methods.

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Numerical Analysis (Chapter 7)	Jacobi & Gauss-Seidel Methods II		R L B	urden &	J D Faires	5	31 / 38

The following are easily verified sufficiency conditions for convergence of the Jacobi and Gauss-Seidel methods.

Theorem

If *A* is strictly diagonally dominant, then for any choice of $\mathbf{x}^{(0)}$, both the Jacobi and Gauss-Seidel methods give sequences $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ that converge to the unique solution of $A\mathbf{x} = \mathbf{b}$.

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Is Gauss-Seidel better than Jacobi?

Is Gauss-Seidel better than Jacobi?

• No general results exist to tell which of the two techniques, Jacobi or Gauss-Seidel, will be most successful for an arbitrary linear system.

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Numerical Analysis (Chapter 7)	Jacobi & Gauss-Seidel Methods II	R L Burden & J D Faires 32 / 38

Is Gauss-Seidel better than Jacobi?

- No general results exist to tell which of the two techniques, Jacobi or Gauss-Seidel, will be most successful for an arbitrary linear system.
- In special cases, however, the answer is known, as is demonstrated in the following theorem.

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Numerical Analysis (Chapter 7)	Jacobi & Gauss-Seidel Methods II		RLB	urden 8	J D Faire	s	32 / 38

(Stein-Rosenberg) Theorem

If $a_{ij} \leq 0$, for each $i \neq j$ and $a_{ii} > 0$, for each i = 1, 2, ..., n, then one and only one of the following statements holds:

- (i) $0 \le \rho(T_g) < \rho(T_j) < 1$
- (ii) $1 < \rho(T_j) < \rho(T_g)$
- (iii) $\rho(T_j) = \rho(T_g) = 0$
- (iv) $\rho(T_j) = \rho(T_g) = 1$

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Numerical Analysis (Chapter 7)	Jacobi & Gauss-Seidel Methods II	R L Burden & J D Faires	33 / 38

(Stein-Rosenberg) Theorem

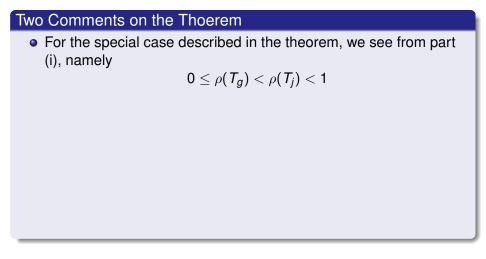
If $a_{ij} \leq 0$, for each $i \neq j$ and $a_{ii} > 0$, for each i = 1, 2, ..., n, then one and only one of the following statements holds:

- (i) $0 \le \rho(T_q) < \rho(T_i) < 1$
- (ii) $1 < \rho(T_j) < \rho(T_g)$
- (iii) $\rho(T_i) = \rho(T_g) = 0$
- (iv) $\rho(T_i) = \rho(T_g) = 1$

For the proof of this result, see pp. 120–127. of

Young, D. M., Iterative solution of large linear systems, Academic Press, New York, 1971, 570 pp.





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Numerical Analysis (Chapter 7)	Jacobi & Gauss-Seidel Methods II	R L Burden & J D Faires	34 / 38

Two Comments on the Thoerem

• For the special case described in the theorem, we see from part (i), namely

 $0 \leq \rho(T_g) < \rho(T_j) < 1$

that when one method gives convergence, then both give convergence,

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Numerical Analysis (Chapter 7)	Jacobi & Gauss-Seidel Methods II	R L Burden & J D Faires	34 / 38

Two Comments on the Thoerem

• For the special case described in the theorem, we see from part (i), namely

 $0 \leq \rho(T_g) < \rho(T_j) < 1$

that when one method gives convergence, then both give convergence, and the Gauss-Seidel method converges faster than the Jacobi method.

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Gauss-Seidel Method Gauss-Seidel Algorithm Convergence Results Interpretation

Convergence of the Jacobi & Gauss-Seidel Methods

Two Comments on the Thoerem

• For the special case described in the theorem, we see from part (i), namely

$$0 \leq \rho(T_g) < \rho(T_j) < 1$$

that when one method gives convergence, then both give convergence, and the Gauss-Seidel method converges faster than the Jacobi method.

• Part (ii), namely

 $1 < \rho(T_j) < \rho(T_g)$

indicates that when one method diverges then both diverge, and the divergence is more pronounced for the Gauss-Seidel method.

Questions?

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Eigenvalues & Eigenvectors: Convergent Matrices

Theorem

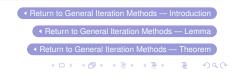
The following statements are equivalent.

- (i) A is a convergent matrix.
- (ii) $\lim_{n\to\infty} ||A^n|| = 0$, for some natural norm.
- (iii) $\lim_{n\to\infty} ||A^n|| = 0$, for all natural norms.

v)
$$\rho(A) < 1$$

(v) $\lim_{n\to\infty} A^n \mathbf{x} = \mathbf{0}$, for every \mathbf{x} .

The proof of this theorem can be found on p. 14 of Issacson, E. and H. B. Keller, Analysis of Numerical Methods, John Wiley & Sons, New York, 1966, 541 pp.



Fixed-Point Theorem

Let $g \in C[a, b]$ be such that $g(x) \in [a, b]$, for all x in [a, b]. Suppose, in addition, that g' exists on (a, b) and that a constant 0 < k < 1 exists with

 $|g'(x)| \le k$, for all $x \in (a, b)$.

Then for any number p_0 in [a, b], the sequence defined by

 $p_n = g(p_{n-1}), \qquad n \ge 1$

converges to the unique fixed point p in [a, b].

Return to the Corrollary to the Fixed-Point Theorem

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Functional (Fixed-Point) Iteration

Corrollary to the Fixed-Point Convergence Result

If g satisfies the hypothesis of the Fixed-Point • Theorem then

$$|p_n-p|\leq \frac{k^n}{1-k}|p_1-p_0|$$

Return to the Corollary to the Convergence Theorem for General Iterative Methods

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