Iterative Techniques in Matrix Algebra

Jacobi & Gauss-Seidel Iterative Techniques II

Numerical Analysis (9th Edition)

R L Burden & J D Faires

Outline

1. The Gauss-Seidel Method

Outline

Outline

Outline

Outline

1. The Gauss-Seidel Method
2. The Gauss-Seidel Algorithm
3. Convergence Results for General Iteration Methods
4. Application to the Jacobi & Gauss-Seidel Methods

The Gauss-Seidel Method

Looking at the Jacobi Method

- A possible improvement to the Jacobi Algorithm can be seen by re-considering

$$x_i^{(k)} = \frac{1}{a_{ii}} \left[\sum_{\substack{j=1 \\ j \neq i}}^{n} \left(-a_{ij} x_j^{(k-1)} \right) + b_i \right], \qquad \text{for } i = 1, 2, \ldots, n$$

The Gauss-Seidel Method

Looking at the Jacobi Method

- A possible improvement to the Jacobi Algorithm can be seen by re-considering

$$x_i^{(k)} = \frac{1}{a_{ii}} \left[\sum_{\substack{j=1 \\ j \neq i}}^{n} \left(-a_{ij} x_j^{(k-1)} \right) + b_i \right], \qquad \text{for } i = 1, 2, \ldots, n$$

- The components of $\mathbf{x}^{(k-1)}$ are used to compute all the components $x_i^{(k)}$ of $\mathbf{x}^{(k)}$.

The Gauss-Seidel Method

Looking at the Jacobi Method

- A possible improvement to the Jacobi Algorithm can be seen by re-considering

$$x_i^{(k)} = \frac{1}{a_{ii}} \left[\sum_{\substack{j=1 \\ j \neq i}}^{n} \left(-a_{ij} x_j^{(k-1)} \right) + b_i \right], \qquad \text{for } i = 1, 2, \ldots, n$$

- The components of $\mathbf{x}^{(k-1)}$ are used to compute all the components $x_i^{(k)}$ of $\mathbf{x}^{(k)}$.
- But, for $i > 1$, the components $x_1^{(k)}, \ldots, x_{i-1}^{(k)}$ of $\mathbf{x}^{(k)}$ have already been computed and are expected to be better approximations to the actual solutions $x_1, \ldots, x_{i-1}$ than are $x_1^{(k-1)}, \ldots, x_{i-1}^{(k-1)}$.

The Gauss-Seidel Method

Instead of using

$$x_i^{(k)} = \frac{1}{a_{ii}} \left[\sum_{\substack{j=1 \\ j \neq i}}^{n} \left(-a_{ij} x_j^{(k-1)} \right) + b_i \right], \qquad \text{for } i = 1, 2, \ldots, n$$

it seems reasonable, then, to compute $x_i^{(k)}$ using these most recently calculated values.

The Gauss-Seidel Method

Instead of using

$$x_i^{(k)} = \frac{1}{a_{ii}} \left[\sum_{\substack{j=1 \\ j \neq i}}^{n} \left(-a_{ij} x_j^{(k-1)} \right) + b_i \right], \qquad \text{for } i = 1, 2, \ldots, n$$

it seems reasonable, then, to compute $x_i^{(k)}$ using these most recently calculated values.

The Gauss-Seidel Iterative Technique

$$x_i^{(k)} = \frac{1}{a_{ii}} \left[-\sum_{j=1}^{i-1} (a_{ij} x_j^{(k)}) - \sum_{j=i+1}^{n} (a_{ij} x_j^{(k-1)}) + b_i \right]$$

for each $i = 1, 2, \ldots, n$.

The Gauss-Seidel Method

Example

Use the Gauss-Seidel iterative technique to find approximate solutions to

$$\begin{aligned} 10x_1 - x_2 + 2x_3 &= 6 \\ -x_1 + 11x_2 - x_3 + 3x_4 &= 25 \\ 2x_1 - x_2 + 10x_3 - x_4 &= -11 \\ 3x_2 - x_3 + 8x_4 &= 15 \end{aligned}$$

The Gauss-Seidel Method

Example

Use the Gauss-Seidel iterative technique to find approximate solutions to

$$\begin{aligned} 10x_1 - x_2 + 2x_3 \qquad &= 6 \\ -x_1 + 11x_2 - x_3 + 3x_4 &= 25 \\ 2x_1 - x_2 + 10x_3 - x_4 &= -11 \\ 3x_2 - x_3 + 8x_4 &= 15 \end{aligned},$$

starting with $\mathbf{x} = (0, 0, 0, 0)^t$

The Gauss-Seidel Method

Example

Use the Gauss-Seidel iterative technique to find approximate solutions to

$$
\begin{aligned}
10x_1 - x_2 + 2x_3 \phantom{{}-3x_4} &= 6\\
-x_1 + 11x_2 - x_3 + 3x_4 &= 25\\
2x_1 - x_2 + 10x_3 - x_4 &= -11\\
3x_2 - x_3 + 8x_4 &= 15
\end{aligned},
$$

starting with $\mathbf{x} = (0, 0, 0, 0)^t$ and iterating until

$$\frac{\|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\|_\infty}{\|\mathbf{x}^{(k)}\|_\infty} < 10^{-3}$$

The Gauss-Seidel Method

Example

Use the Gauss-Seidel iterative technique to find approximate solutions to

$$\begin{aligned} 10x_1 - x_2 + 2x_3 \qquad &= 6 \\ -x_1 + 11x_2 - x_3 + 3x_4 &= 25 \\ 2x_1 - x_2 + 10x_3 - x_4 &= -11, \\ 3x_2 - x_3 + 8x_4 &= 15 \end{aligned}$$

starting with $\mathbf{x} = (0, 0, 0, 0)^t$ and iterating until

$$\frac{\|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\|_\infty}{\|\mathbf{x}^{(k)}\|_\infty} < 10^{-3}$$

Note: The solution $\mathbf{x} = (1,\ 2,\ -1,\ 1)^t$ was approximated by Jacobi's method in an earlier example.

The Gauss-Seidel Method

Solution (1/3)

For the Gauss-Seidel method we write the system, for each $k = 1, 2, \ldots$ as

$$
\begin{aligned}
x_1^{(k)} &= \qquad\qquad\quad \frac{1}{10}x_2^{(k-1)} - \frac{1}{5}x_3^{(k-1)} \qquad\qquad\quad + \frac{3}{5} \\
x_2^{(k)} &= \frac{1}{11}x_1^{(k)} \qquad\qquad\quad + \frac{1}{11}x_3^{(k-1)} - \frac{3}{11}x_4^{(k-1)} + \frac{25}{11} \\
x_3^{(k)} &= -\frac{1}{5}x_1^{(k)} + \frac{1}{10}x_2^{(k)} \qquad\qquad\quad + \frac{1}{10}x_4^{(k-1)} - \frac{11}{10} \\
x_4^{(k)} &= \qquad\qquad - \frac{3}{8}x_2^{(k)} + \frac{1}{8}x_3^{(k)} \qquad\qquad\quad + \frac{15}{8}
\end{aligned}
$$

The Gauss-Seidel Method

Solution (2/3)

When $\mathbf{x}^{(0)} = (0, 0, 0, 0)^t$, we have
$\mathbf{x}^{(1)} = (0.6000, 2.3272, -0.9873, 0.8789)^t$.

The Gauss-Seidel Method

Solution (2/3)

When $\mathbf{x}^{(0)} = (0, 0, 0, 0)^t$, we have $\mathbf{x}^{(1)} = (0.6000, 2.3272, -0.9873, 0.8789)^t$. Subsequent iterations give the values in the following table:

k	0	1	2	3	4	5
$x_1^{(k)}$	0.0000	0.6000	1.030	1.0065	1.0009	1.0001
$x_2^{(k)}$	0.0000	2.3272	2.037	2.0036	2.0003	2.0000
$x_3^{(k)}$	0.0000	-0.9873	-1.014	-1.0025	-1.0003	-1.0000
$x_4^{(k)}$	0.0000	0.8789	0.984	0.9983	0.9999	1.0000

The Gauss-Seidel Method

Solution (3/3)

Because

$$\frac{\|\mathbf{x}^{(5)} - \mathbf{x}^{(4)}\|_\infty}{\|\mathbf{x}^{(5)}\|_\infty} = \frac{0.0008}{2.000} = 4 \times 10^{-4}$$

$\mathbf{x}^{(5)}$ is accepted as a reasonable approximation to the solution.

The Gauss-Seidel Method

Solution (3/3)

Because

$$\frac{\|\mathbf{x}^{(5)} - \mathbf{x}^{(4)}\|_\infty}{\|\mathbf{x}^{(5)}\|_\infty} = \frac{0.0008}{2.000} = 4 \times 10^{-4}$$

$\mathbf{x}^{(5)}$ is accepted as a reasonable approximation to the solution.

Note that, in an earlier example, Jacobi's method required twice as many iterations for the same accuracy.

The Gauss-Seidel Method: Matrix Form

Re-Writing the Equations

To write the Gauss-Seidel method in matrix form,

The Gauss-Seidel Method: Matrix Form

Re-Writing the Equations

To write the Gauss-Seidel method in matrix form, multiply both sides of

$$x_i^{(k)} = \frac{1}{a_{ii}} \left[-\sum_{j=1}^{i-1} (a_{ij} x_j^{(k)}) - \sum_{j=i+1}^{n} (a_{ij} x_j^{(k-1)}) + b_i \right]$$

by a_{ii} and collect all kth iterate terms,

The Gauss-Seidel Method: Matrix Form

Re-Writing the Equations

To write the Gauss-Seidel method in matrix form, multiply both sides of

$$x_i^{(k)} = \frac{1}{a_{ii}} \left[-\sum_{j=1}^{i-1} (a_{ij} x_j^{(k)}) - \sum_{j=i+1}^{n} (a_{ij} x_j^{(k-1)}) + b_i \right]$$

by a_{ii} and collect all kth iterate terms, to give

$$a_{i1} x_1^{(k)} + a_{i2} x_2^{(k)} + \cdots + a_{ii} x_i^{(k)} = -a_{i,i+1} x_{i+1}^{(k-1)} - \cdots - a_{in} x_n^{(k-1)} + b_i$$

for each $i = 1, 2, \ldots, n$.

The Gauss-Seidel Method: Matrix Form

Re-Writing the Equations (Cont'd)

Writing all n equations gives

$$\begin{array}{lllll}
a_{11}x_1^{(k)} & & & = & -a_{12}x_2^{(k-1)} - a_{13}x_3^{(k-1)} - \cdots - a_{1n}x_n^{(k-1)} + b_1 \\
a_{21}x_1^{(k)} & + & a_{22}x_2^{(k)} & = & -a_{23}x_3^{(k-1)} - \cdots - a_{2n}x_n^{(k-1)} + b_2 \\
\vdots & & & & \\
a_{n1}x_1^{(k)} & + & a_{n2}x_2^{(k)} + \cdots + a_{nn}x_n^{(k)} & = & b_n
\end{array}$$

The Gauss-Seidel Method: Matrix Form

Re-Writing the Equations (Cont'd)

Writing all *n* equations gives

$$
\begin{array}{lcl}
a_{11}x_1^{(k)} & = & -a_{12}x_2^{(k-1)} - a_{13}x_3^{(k-1)} - \cdots - a_{1n}x_n^{(k-1)} + b_1 \\
a_{21}x_1^{(k)} + a_{22}x_2^{(k)} & = & -a_{23}x_3^{(k-1)} - \cdots - a_{2n}x_n^{(k-1)} + b_2 \\
\vdots & & \\
a_{n1}x_1^{(k)} + a_{n2}x_2^{(k)} + \cdots + a_{nn}x_n^{(k)} & = & b_n
\end{array}
$$

With the definitions of *D*, *L*, and *U* given previously, we have the Gauss-Seidel method represented by

$$(D-L)\mathbf{x}^{(k)} = U\mathbf{x}^{(k-1)} + \mathbf{b}$$

The Gauss-Seidel Method: Matrix Form

$$(D - L)\mathbf{x}^{(k)} = U\mathbf{x}^{(k-1)} + \mathbf{b}$$

Re-Writing the Equations (Cont'd)

Solving for $\mathbf{x}^{(k)}$ finally gives

$$\mathbf{x}^{(k)} = (D - L)^{-1}U\mathbf{x}^{(k-1)} + (D - L)^{-1}\mathbf{b}, \quad \text{for each } k = 1, 2, \ldots$$

The Gauss-Seidel Method: Matrix Form

$$(D - L)\mathbf{x}^{(k)} = U\mathbf{x}^{(k-1)} + \mathbf{b}$$

Re-Writing the Equations (Cont'd)

Solving for $\mathbf{x}^{(k)}$ finally gives

$$\mathbf{x}^{(k)} = (D - L)^{-1}U\mathbf{x}^{(k-1)} + (D - L)^{-1}\mathbf{b}, \quad \text{for each } k = 1, 2, \ldots$$

Letting $T_g = (D - L)^{-1}U$ and $\mathbf{c}_g = (D - L)^{-1}\mathbf{b}$, gives the Gauss-Seidel technique the form

$$\mathbf{x}^{(k)} = T_g\mathbf{x}^{(k-1)} + \mathbf{c}_g$$

The Gauss-Seidel Method: Matrix Form

$$(D-L)\mathbf{x}^{(k)} = U\mathbf{x}^{(k-1)} + \mathbf{b}$$

Re-Writing the Equations (Cont'd)

Solving for $\mathbf{x}^{(k)}$ finally gives

$$\mathbf{x}^{(k)} = (D-L)^{-1}U\mathbf{x}^{(k-1)} + (D-L)^{-1}\mathbf{b}, \quad \text{for each } k = 1, 2, \ldots$$

Letting $T_g = (D-L)^{-1}U$ and $\mathbf{c}_g = (D-L)^{-1}\mathbf{b}$, gives the Gauss-Seidel technique the form

$$\mathbf{x}^{(k)} = T_g\mathbf{x}^{(k-1)} + \mathbf{c}_g$$

For the lower-triangular matrix $D-L$ to be nonsingular, it is necessary and sufficient that $a_{ii} \neq 0$, for each $i = 1, 2, \ldots, n$.

Outline

Gauss-Seidel Iterative Algorithm (1/2)

To solve $A\mathbf{x} = \mathbf{b}$ given an initial approximation $\mathbf{x}^{(0)}$:

Gauss-Seidel Iterative Algorithm (1/2)

To solve $A\mathbf{x} = \mathbf{b}$ given an initial approximation $\mathbf{x}^{(0)}$:

INPUT the number of equations and unknowns n;
the entries a_{ij}, $1 \leq i, j \leq n$ of the matrix A;
the entries b_i, $1 \leq i \leq n$ of $\mathbf{b}$;
the entries XO_i, $1 \leq i \leq n$ of $\mathbf{XO} = \mathbf{x}^{(0)}$;
tolerance TOL;
maximum number of iterations N.

Gauss-Seidel Iterative Algorithm (1/2)

To solve $A\mathbf{x} = \mathbf{b}$ given an initial approximation $\mathbf{x}^{(0)}$:

INPUT the number of equations and unknowns n;
the entries a_{ij}, $1 \leq i, j \leq n$ of the matrix A;
the entries b_i, $1 \leq i \leq n$ of $\mathbf{b}$;
the entries XO_i, $1 \leq i \leq n$ of $\mathbf{XO} = \mathbf{x}^{(0)}$;
tolerance TOL;
maximum number of iterations N.

OUTPUT the approximate solution $x_1, \ldots, x_n$ or a message that the number of iterations was exceeded.

Gauss-Seidel Iterative Algorithm (2/2)

Step 1 Set $k = 1$

Step 2 While $(k \leq N)$ do Steps 3–6:

Gauss-Seidel Iterative Algorithm (2/2)

Step 1 Set $k = 1$

Step 2 While ($k \leq N$) do Steps 3–6:

Step 3 For $i = 1, \ldots, n$

$$\text{set } x_i = \frac{1}{a_{ii}} \left[-\sum_{j=1}^{i-1} a_{ij} x_j - \sum_{j=i+1}^{n} a_{ij} XO_j + b_i \right]$$

Gauss-Seidel Iterative Algorithm (2/2)

Step 1 Set $k = 1$

Step 2 While ($k \leq N$) do Steps 3–6:

Step 3 For $i = 1, \ldots, n$

$$\text{set } x_i = \frac{1}{a_{ii}} \left[-\sum_{j=1}^{i-1} a_{ij} x_j - \sum_{j=i+1}^{n} a_{ij} XO_j + b_i \right]$$

Step 4 If $||\mathbf{x} - \mathbf{XO}|| < TOL$ then OUTPUT $(x_1, \ldots, x_n)$

(*The procedure was successful*)

STOP

Gauss-Seidel Iterative Algorithm (2/2)

Step 1 Set $k = 1$

Step 2 While ($k \leq N$) do Steps 3–6:

Step 3 For $i = 1, \ldots, n$

$$\text{set } x_i = \frac{1}{a_{ii}} \left[-\sum_{j=1}^{i-1} a_{ij} x_j - \sum_{j=i+1}^{n} a_{ij} XO_j + b_i \right]$$

Step 4 If $||\mathbf{x} - \mathbf{XO}|| < TOL$ then OUTPUT $(x_1, \ldots, x_n)$

(*The procedure was successful*)

STOP

Step 5 Set $k = k + 1$

Gauss-Seidel Iterative Algorithm (2/2)

Step 1 Set $k = 1$

Step 2 While ($k \leq N$) do Steps 3–6:

Step 3 For $i = 1, \ldots, n$

$$\text{set } x_i = \frac{1}{a_{ii}} \left[-\sum_{j=1}^{i-1} a_{ij} x_j - \sum_{j=i+1}^{n} a_{ij} XO_j + b_i \right]$$

Step 4 If $||\mathbf{x} - \mathbf{XO}|| < TOL$ then OUTPUT $(x_1, \ldots, x_n)$

(*The procedure was successful*)

STOP

Step 5 Set $k = k + 1$

Step 6 For $i = 1, \ldots, n$ set $XO_i = x_i$

Gauss-Seidel Iterative Algorithm (2/2)

Step 1 Set $k = 1$

Step 2 While ($k \leq N$) do Steps 3–6:

Step 3 For $i = 1, \ldots, n$

$$\text{set } x_i = \frac{1}{a_{ii}} \left[-\sum_{j=1}^{i-1} a_{ij} x_j - \sum_{j=i+1}^{n} a_{ij} XO_j + b_i \right]$$

Step 4 If $||\mathbf{x} - \mathbf{XO}|| < TOL$ then OUTPUT $(x_1, \ldots, x_n)$
(*The procedure was successful*)
STOP

Step 5 Set $k = k + 1$

Step 6 For $i = 1, \ldots, n$ set $XO_i = x_i$

Step 7 OUTPUT ('Maximum number of iterations exceeded')
STOP (*The procedure was unsuccessful*)

Gauss-Seidel Iterative Algorithm

Comments on the Algorithm

- Step 3 of the algorithm requires that $a_{ii} \neq 0$, for each $i = 1, 2, \ldots, n$.

Gauss-Seidel Iterative Algorithm

Comments on the Algorithm

- Step 3 of the algorithm requires that $a_{ii} \neq 0$, for each $i = 1, 2, \ldots, n$. If one of the a_{ii} entries is 0 and the system is nonsingular, a reordering of the equations can be performed so that no $a_{ii} = 0$.

Gauss-Seidel Iterative Algorithm

Comments on the Algorithm

- Step 3 of the algorithm requires that $a_{ii} \neq 0$, for each $i = 1, 2, \ldots, n$. If one of the a_{ii} entries is 0 and the system is nonsingular, a reordering of the equations can be performed so that no $a_{ii} = 0$.
- To speed convergence, the equations should be arranged so that a_{ii} is as large as possible.

Gauss-Seidel Iterative Algorithm

Comments on the Algorithm

- Step 3 of the algorithm requires that $a_{ii} \neq 0$, for each $i = 1, 2, \ldots, n$. If one of the a_{ii} entries is 0 and the system is nonsingular, a reordering of the equations can be performed so that no $a_{ii} = 0$.
- To speed convergence, the equations should be arranged so that a_{ii} is as large as possible.
- Another possible stopping criterion in Step 4 is to iterate until

$$\frac{\|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\|}{\|\mathbf{x}^{(k)}\|}$$

  is smaller than some prescribed tolerance.

Gauss-Seidel Iterative Algorithm

Comments on the Algorithm

- Step 3 of the algorithm requires that $a_{ii} \neq 0$, for each $i = 1, 2, \ldots, n$. If one of the a_{ii} entries is 0 and the system is nonsingular, a reordering of the equations can be performed so that no $a_{ii} = 0$.
- To speed convergence, the equations should be arranged so that a_{ii} is as large as possible.
- Another possible stopping criterion in Step 4 is to iterate until

$$\frac{\|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\|}{\|\mathbf{x}^{(k)}\|}$$

  is smaller than some prescribed tolerance.
- For this purpose, any convenient norm can be used, the usual being the l_∞ norm.

Outline

Convergence Results for General Iteration Methods

Introduction

- To study the convergence of general iteration techniques, we need to analyze the formula

$$\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c}, \quad \text{for each } k = 1, 2, \ldots$$

where $\mathbf{x}^{(0)}$ is arbitrary.
- The following lemma and the earlier Theorem on convergent matrices provide the key for this study.

Convergence Results for General Iteration Methods

Lemma

If the spectral radius satisfies $\rho(T) < 1$, then $(I - T)^{-1}$ exists, and

$$(I - T)^{-1} = I + T + T^2 + \cdots = \sum_{j=0}^{\infty} T^j$$

Convergence Results for General Iteration Methods

Lemma

If the spectral radius satisfies $\rho(T) < 1$, then $(I - T)^{-1}$ exists, and

$$(I - T)^{-1} = I + T + T^2 + \cdots = \sum_{j=0}^{\infty} T^j$$

Proof (1/2)

- Because $T\mathbf{x} = \lambda\mathbf{x}$ is true precisely when $(I - T)\mathbf{x} = (1 - \lambda)\mathbf{x}$, we have λ as an eigenvalue of T precisely when $1 - \lambda$ is an eigenvalue of $I - T$.

Convergence Results for General Iteration Methods

Lemma

If the spectral radius satisfies $\rho(T) < 1$, then $(I - T)^{-1}$ exists, and

$$(I - T)^{-1} = I + T + T^2 + \cdots = \sum_{j=0}^{\infty} T^j$$

Proof (1/2)

- Because $T\mathbf{x} = \lambda\mathbf{x}$ is true precisely when $(I - T)\mathbf{x} = (1 - \lambda)\mathbf{x}$, we have λ as an eigenvalue of T precisely when $1 - \lambda$ is an eigenvalue of $I - T$.
- But $|\lambda| \leq \rho(T) < 1$, so $\lambda = 1$ is not an eigenvalue of T, and 0 cannot be an eigenvalue of $I - T$.

Convergence Results for General Iteration Methods

Lemma

If the spectral radius satisfies $\rho(T) < 1$, then $(I - T)^{-1}$ exists, and

$$(I - T)^{-1} = I + T + T^2 + \cdots = \sum_{j=0}^{\infty} T^j$$

Proof (1/2)

- Because $T\mathbf{x} = \lambda\mathbf{x}$ is true precisely when $(I - T)\mathbf{x} = (1 - \lambda)\mathbf{x}$, we have λ as an eigenvalue of T precisely when $1 - \lambda$ is an eigenvalue of $I - T$.
- But $|\lambda| \le \rho(T) < 1$, so $\lambda = 1$ is not an eigenvalue of T, and 0 cannot be an eigenvalue of $I - T$.
- Hence, $(I - T)^{-1}$ exists.

Convergence Results for General Iteration Methods

Proof (2/2)

Let

$$S_m = I + T + T^2 + \cdots + T^m$$

Convergence Results for General Iteration Methods

Proof (2/2)

Let

$$S_m = I + T + T^2 + \cdots + T^m$$

Then

$$(I - T)S_m = (1 + T + T^2 + \cdots + T^m) - (T + T^2 + \cdots + T^{m+1}) = I - T^{m+1}$$

Convergence Results for General Iteration Methods

Proof (2/2)

Let

$$S_m = I + T + T^2 + \cdots + T^m$$

Then

$$(I - T)S_m = (1 + T + T^2 + \cdots + T^m) - (T + T^2 + \cdots + T^{m+1}) = I - T^{m+1}$$

and, since T is convergent, the ▸ Theorem on convergent matrices implies that

$$\lim_{m\to\infty}(I - T)S_m = \lim_{m\to\infty}(I - T^{m+1}) = I$$

Convergence Results for General Iteration Methods

Proof (2/2)

Let

$$S_m = I + T + T^2 + \cdots + T^m$$

Then

$$(I - T)S_m = (1 + T + T^2 + \cdots + T^m) - (T + T^2 + \cdots + T^{m+1}) = I - T^{m+1}$$

and, since T is convergent, the ▸ Theorem on convergent matrices implies that

$$\lim_{m\to\infty}(I - T)S_m = \lim_{m\to\infty}(I - T^{m+1}) = I$$

Thus, $(I - T)^{-1} = \lim_{m\to\infty} S_m = I + T + T^2 + \cdots = \sum_{j=0}^{\infty} T^j$

Convergence Results for General Iteration Methods

Theorem

For any $\mathbf{x}^{(0)} \in \mathbb{R}^n$, the sequence $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ defined by

$$\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c}, \quad \text{for each } k \geq 1$$

converges to the unique solution of

$$\mathbf{x} = T\mathbf{x} + \mathbf{c}$$

if and only if $\rho(T) < 1$.

Convergence Results for General Iteration Methods

Proof (1/5)

First assume that $\rho(T) < 1$.

Convergence Results for General Iteration Methods

Proof (1/5)

First assume that $\rho(T) < 1$. Then,

$$\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c}$$

Convergence Results for General Iteration Methods

Proof (1/5)

First assume that $\rho(T) < 1$. Then,

$$\begin{aligned} \mathbf{x}^{(k)} &= T\mathbf{x}^{(k-1)} + \mathbf{c} \\ &= T(T\mathbf{x}^{(k-2)} + \mathbf{c}) + \mathbf{c} \end{aligned}$$

Convergence Results for General Iteration Methods

Proof (1/5)

First assume that $\rho(T) < 1$. Then,

$$\begin{aligned} \mathbf{x}^{(k)} &= T\mathbf{x}^{(k-1)} + \mathbf{c} \\ &= T(T\mathbf{x}^{(k-2)} + \mathbf{c}) + \mathbf{c} \\ &= T^2\mathbf{x}^{(k-2)} + (T + I)\mathbf{c} \end{aligned}$$

Convergence Results for General Iteration Methods

Proof (1/5)

First assume that $\rho(T) < 1$. Then,

$$\begin{aligned}
\mathbf{x}^{(k)} &= T\mathbf{x}^{(k-1)} + \mathbf{c} \\
&= T(T\mathbf{x}^{(k-2)} + \mathbf{c}) + \mathbf{c} \\
&= T^2\mathbf{x}^{(k-2)} + (T + I)\mathbf{c} \\
&\vdots \\
&= T^k\mathbf{x}^{(0)} + (T^{k-1} + \cdots + T + I)\mathbf{c}
\end{aligned}$$

Convergence Results for General Iteration Methods

Proof (1/5)

First assume that $\rho(T) < 1$. Then,

$$\begin{aligned}
\mathbf{x}^{(k)} &= T\mathbf{x}^{(k-1)} + \mathbf{c} \\
&= T(T\mathbf{x}^{(k-2)} + \mathbf{c}) + \mathbf{c} \\
&= T^2\mathbf{x}^{(k-2)} + (T + I)\mathbf{c} \\
&\vdots \\
&= T^k\mathbf{x}^{(0)} + (T^{k-1} + \cdots + T + I)\mathbf{c}
\end{aligned}$$

Because $\rho(T) < 1$, the Theorem on convergent matrices implies that T is convergent, and

$$\lim_{k\to\infty} T^k\mathbf{x}^{(0)} = \mathbf{0}$$

Convergence Results for General Iteration Methods

Proof (2/5)

The previous lemma implies that

$$\lim_{k\to\infty} \mathbf{x}^{(k)} = \lim_{k\to\infty} T^k \mathbf{x}^{(0)} + \left(\sum_{j=0}^{\infty} T^j \right) \mathbf{c}$$

Convergence Results for General Iteration Methods

Proof (2/5)

The previous lemma implies that

$$\begin{aligned}\lim_{k\to\infty} \mathbf{x}^{(k)} &= \lim_{k\to\infty} T^k \mathbf{x}^{(0)} + \left(\sum_{j=0}^{\infty} T^j\right)\mathbf{c} \\ &= \mathbf{0} + (I - T)^{-1}\mathbf{c}\end{aligned}$$

Convergence Results for General Iteration Methods

Proof (2/5)

The previous lemma implies that

$$
\begin{aligned}
\lim_{k\to\infty} \mathbf{x}^{(k)} &= \lim_{k\to\infty} T^k \mathbf{x}^{(0)} + \left(\sum_{j=0}^{\infty} T^j \right) \mathbf{c} \\
&= \mathbf{0} + (I - T)^{-1} \mathbf{c} \\
&= (I - T)^{-1} \mathbf{c}
\end{aligned}
$$

Convergence Results for General Iteration Methods

Proof (2/5)

The previous lemma implies that

$$\begin{aligned}\lim_{k\to\infty} \mathbf{x}^{(k)} &= \lim_{k\to\infty} T^k \mathbf{x}^{(0)} + \left(\sum_{j=0}^{\infty} T^j\right)\mathbf{c} \\ &= \mathbf{0} + (I-T)^{-1}\mathbf{c} \\ &= (I-T)^{-1}\mathbf{c}\end{aligned}$$

Hence, the sequence $\{\mathbf{x}^{(k)}\}$ converges to the vector $\mathbf{x} \equiv (I-T)^{-1}\mathbf{c}$ and $\mathbf{x} = T\mathbf{x} + \mathbf{c}$.

Convergence Results for General Iteration Methods

Proof (3/5)

- To prove the converse, we will show that for any $\mathbf{z} \in \mathbb{R}^n$, we have $\lim_{k\to\infty} T^k\mathbf{z} = \mathbf{0}$.

Convergence Results for General Iteration Methods

Proof (3/5)

- To prove the converse, we will show that for any $\mathbf{z} \in \mathbb{R}^n$, we have $\lim_{k\to\infty} T^k\mathbf{z} = \mathbf{0}$.
- Again, by the theorem on convergent matrices, this is equivalent to $\rho(T) < 1$.

Convergence Results for General Iteration Methods

Proof (3/5)

- To prove the converse, we will show that for any $\mathbf{z} \in \mathbb{R}^n$, we have $\lim_{k\to\infty} T^k\mathbf{z} = \mathbf{0}$.
- Again, by the theorem on convergent matrices, this is equivalent to $\rho(T) < 1$.
- Let $\mathbf{z}$ be an arbitrary vector, and $\mathbf{x}$ be the unique solution to $\mathbf{x} = T\mathbf{x} + \mathbf{c}$.

Convergence Results for General Iteration Methods

Proof (3/5)

- To prove the converse, we will show that for any $\mathbf{z} \in \mathbb{R}^n$, we have $\lim_{k\to\infty} T^k\mathbf{z} = \mathbf{0}$.
- Again, by the theorem on convergent matrices, this is equivalent to $\rho(T) < 1$.
- Let $\mathbf{z}$ be an arbitrary vector, and $\mathbf{x}$ be the unique solution to $\mathbf{x} = T\mathbf{x} + \mathbf{c}$.
- Define $\mathbf{x}^{(0)} = \mathbf{x} - \mathbf{z}$, and, for $k \geq 1$, $\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c}$.

Convergence Results for General Iteration Methods

Proof (3/5)

- To prove the converse, we will show that for any $\mathbf{z} \in \mathbb{R}^n$, we have $\lim_{k\to\infty} T^k\mathbf{z} = \mathbf{0}$.
- Again, by the theorem on convergent matrices, this is equivalent to $\rho(T) < 1$.
- Let $\mathbf{z}$ be an arbitrary vector, and $\mathbf{x}$ be the unique solution to $\mathbf{x} = T\mathbf{x} + \mathbf{c}$.
- Define $\mathbf{x}^{(0)} = \mathbf{x} - \mathbf{z}$, and, for $k \geq 1$, $\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c}$.
- Then $\{\mathbf{x}^{(k)}\}$ converges to $\mathbf{x}$.

Convergence Results for General Iteration Methods

Proof (4/5)

Also,

$$\mathbf{x} - \mathbf{x}^{(k)} = (T\mathbf{x} + \mathbf{c}) - \left(T\mathbf{x}^{(k-1)} + \mathbf{c}\right) = T\left(\mathbf{x} - \mathbf{x}^{(k-1)}\right)$$

Convergence Results for General Iteration Methods

Proof (4/5)

Also,

$$\mathbf{x} - \mathbf{x}^{(k)} = (T\mathbf{x} + \mathbf{c}) - \left(T\mathbf{x}^{(k-1)} + \mathbf{c}\right) = T\left(\mathbf{x} - \mathbf{x}^{(k-1)}\right)$$

so

$$\mathbf{x} - \mathbf{x}^{(k)} \quad = \quad T\left(\mathbf{x} - \mathbf{x}^{(k-1)}\right)$$

Convergence Results for General Iteration Methods

Proof (4/5)

Also,

$$\mathbf{x} - \mathbf{x}^{(k)} = (T\mathbf{x} + \mathbf{c}) - \left(T\mathbf{x}^{(k-1)} + \mathbf{c}\right) = T\left(\mathbf{x} - \mathbf{x}^{(k-1)}\right)$$

so

$$\begin{aligned} \mathbf{x} - \mathbf{x}^{(k)} &= T\left(\mathbf{x} - \mathbf{x}^{(k-1)}\right) \\ &= T^2\left(\mathbf{x} - \mathbf{x}^{(k-2)}\right) \end{aligned}$$

Convergence Results for General Iteration Methods

Proof (4/5)

Also,

$$\mathbf{x} - \mathbf{x}^{(k)} = (T\mathbf{x} + \mathbf{c}) - \left(T\mathbf{x}^{(k-1)} + \mathbf{c}\right) = T\left(\mathbf{x} - \mathbf{x}^{(k-1)}\right)$$

so

$$\begin{aligned} \mathbf{x} - \mathbf{x}^{(k)} &= T\left(\mathbf{x} - \mathbf{x}^{(k-1)}\right) \\ &= T^2\left(\mathbf{x} - \mathbf{x}^{(k-2)}\right) \\ &= \vdots \end{aligned}$$

Convergence Results for General Iteration Methods

Proof (4/5)

Also,

$$\mathbf{x} - \mathbf{x}^{(k)} = (T\mathbf{x} + \mathbf{c}) - \left(T\mathbf{x}^{(k-1)} + \mathbf{c}\right) = T\left(\mathbf{x} - \mathbf{x}^{(k-1)}\right)$$

so

$$\begin{aligned}
\mathbf{x} - \mathbf{x}^{(k)} &= T\left(\mathbf{x} - \mathbf{x}^{(k-1)}\right) \\
&= T^2\left(\mathbf{x} - \mathbf{x}^{(k-2)}\right) \\
&= \vdots \\
&= T^k\left(\mathbf{x} - \mathbf{x}^{(0)}\right)
\end{aligned}$$

Convergence Results for General Iteration Methods

Proof (4/5)

Also,

$$\mathbf{x} - \mathbf{x}^{(k)} = (T\mathbf{x} + \mathbf{c}) - \left(T\mathbf{x}^{(k-1)} + \mathbf{c}\right) = T\left(\mathbf{x} - \mathbf{x}^{(k-1)}\right)$$

so

$$\begin{aligned} \mathbf{x} - \mathbf{x}^{(k)} &= T\left(\mathbf{x} - \mathbf{x}^{(k-1)}\right) \\ &= T^2\left(\mathbf{x} - \mathbf{x}^{(k-2)}\right) \\ &= \vdots \\ &= T^k\left(\mathbf{x} - \mathbf{x}^{(0)}\right) \\ &= T^k\mathbf{z} \end{aligned}$$

Convergence Results for General Iteration Methods

Proof (5/5)

- Hence

$$\lim_{k\to\infty} T^k \mathbf{z} \quad = \quad \lim_{k\to\infty} T^k \left(\mathbf{x} - \mathbf{x}^{(0)}\right)$$

Convergence Results for General Iteration Methods

Proof (5/5)

- Hence

$$\begin{aligned} \lim_{k\to\infty} T^k \mathbf{z} &= \lim_{k\to\infty} T^k \left(\mathbf{x} - \mathbf{x}^{(0)} \right) \\ &= \lim_{k\to\infty} \left(\mathbf{x} - \mathbf{x}^{(k)} \right) \end{aligned}$$

Convergence Results for General Iteration Methods

Proof (5/5)

- Hence

$$\begin{aligned}
\lim_{k\to\infty} T^k \mathbf{z} &= \lim_{k\to\infty} T^k \left(\mathbf{x} - \mathbf{x}^{(0)}\right) \\
&= \lim_{k\to\infty} \left(\mathbf{x} - \mathbf{x}^{(k)}\right) \\
&= \mathbf{0}
\end{aligned}$$

Convergence Results for General Iteration Methods

Proof (5/5)

- Hence

$$\begin{aligned} \lim_{k\to\infty} T^k \mathbf{z} &= \lim_{k\to\infty} T^k \left(\mathbf{x} - \mathbf{x}^{(0)}\right) \\ &= \lim_{k\to\infty} \left(\mathbf{x} - \mathbf{x}^{(k)}\right) \\ &= \mathbf{0} \end{aligned}$$

- But $\mathbf{z} \in \mathbb{R}^n$ was arbitrary, so by the theorem on convergent matrices, T is convergent and $\rho(T) < 1$.

Convergence Results for General Iteration Methods

Corollary

$\|T\| < 1$ for any natural matrix norm and $\mathbf{c}$ is a given vector, then the sequence $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ defined by

$$\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c}$$

converges, for any $\mathbf{x}^{(0)} \in \mathbb{R}^n$, to a vector $\mathbf{x} \in \mathbb{R}^n$, with $\mathbf{x} = T\mathbf{x} + \mathbf{c}$, and the following error bounds hold:

Convergence Results for General Iteration Methods

Corollary

$\|T\| < 1$ for any natural matrix norm and $\mathbf{c}$ is a given vector, then the sequence $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ defined by

$$\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c}$$

converges, for any $\mathbf{x}^{(0)} \in \mathbb{R}^n$, to a vector $\mathbf{x} \in \mathbb{R}^n$, with $\mathbf{x} = T\mathbf{x} + \mathbf{c}$, and the following error bounds hold:

(i) $\|\mathbf{x} - \mathbf{x}^{(k)}\| \leq \|T\|^k \|\mathbf{x}^{(0)} - \mathbf{x}\|$

Convergence Results for General Iteration Methods

Corollary

$\|T\| < 1$ for any natural matrix norm and $\mathbf{c}$ is a given vector, then the sequence $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ defined by

$$\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c}$$

converges, for any $\mathbf{x}^{(0)} \in \mathbb{R}^n$, to a vector $\mathbf{x} \in \mathbb{R}^n$, with $\mathbf{x} = T\mathbf{x} + \mathbf{c}$, and the following error bounds hold:

(i) $\|\mathbf{x} - \mathbf{x}^{(k)}\| \leq \|T\|^k \|\mathbf{x}^{(0)} - \mathbf{x}\|$

(ii) $\|\mathbf{x} - \mathbf{x}^{(k)}\| \leq \frac{\|T\|^k}{1-\|T\|} \|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\|$

Convergence Results for General Iteration Methods

Corollary

$\|T\| < 1$ for any natural matrix norm and $\mathbf{c}$ is a given vector, then the sequence $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ defined by

$$\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c}$$

converges, for any $\mathbf{x}^{(0)} \in \mathbb{R}^n$, to a vector $\mathbf{x} \in \mathbb{R}^n$, with $\mathbf{x} = T\mathbf{x} + \mathbf{c}$, and the following error bounds hold:

(i) $\|\mathbf{x} - \mathbf{x}^{(k)}\| \leq \|T\|^k \|\mathbf{x}^{(0)} - \mathbf{x}\|$

(ii) $\|\mathbf{x} - \mathbf{x}^{(k)}\| \leq \frac{\|T\|^k}{1-\|T\|} \|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\|$

The proof of the following corollary is similar to that for the Corollary to the Fixed-Point Theorem for a single nonlinear equation.

Outline

Convergence of the Jacobi & Gauss-Seidel Methods

Using the Matrix Formulations

We have seen that the Jacobi and Gauss-Seidel iterative techniques can be written

$$\begin{aligned} \mathbf{x}^{(k)} &= T_j\mathbf{x}^{(k-1)} + \mathbf{c}_j \quad \text{and} \\ \mathbf{x}^{(k)} &= T_g\mathbf{x}^{(k-1)} + \mathbf{c}_g \end{aligned}$$

Convergence of the Jacobi & Gauss-Seidel Methods

Using the Matrix Formulations

We have seen that the Jacobi and Gauss-Seidel iterative techniques can be written

$$\begin{aligned}\mathbf{x}^{(k)} &= T_j\mathbf{x}^{(k-1)} + \mathbf{c}_j \quad \text{and} \\ \mathbf{x}^{(k)} &= T_g\mathbf{x}^{(k-1)} + \mathbf{c}_g\end{aligned}$$

using the matrices

$$T_j = D^{-1}(L+U) \quad \text{and} \quad T_g = (D-L)^{-1}U$$

respectively.

Convergence of the Jacobi & Gauss-Seidel Methods

Using the Matrix Formulations

We have seen that the Jacobi and Gauss-Seidel iterative techniques can be written

$$\begin{aligned} \mathbf{x}^{(k)} &= T_j\mathbf{x}^{(k-1)} + \mathbf{c}_j \quad \text{and} \\ \mathbf{x}^{(k)} &= T_g\mathbf{x}^{(k-1)} + \mathbf{c}_g \end{aligned}$$

using the matrices

$$T_j = D^{-1}(L+U) \quad \text{and} \quad T_g = (D-L)^{-1}U$$

respectively. If $\rho(T_j)$ or $\rho(T_g)$ is less than 1, then the corresponding sequence $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ will converge to the solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$.

Convergence of the Jacobi & Gauss-Seidel Methods

Example

For example, the Jacobi method has

$$\mathbf{x}^{(k)} = D^{-1}(L + U)\mathbf{x}^{(k-1)} + D^{-1}\mathbf{b},$$

Convergence of the Jacobi & Gauss-Seidel Methods

Example

For example, the Jacobi method has

$$\mathbf{x}^{(k)} = D^{-1}(L+U)\mathbf{x}^{(k-1)} + D^{-1}\mathbf{b},$$

and, if $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ converges to $\mathbf{x}$,

Convergence of the Jacobi & Gauss-Seidel Methods

Example

For example, the Jacobi method has

$$\mathbf{x}^{(k)} = D^{-1}(L+U)\mathbf{x}^{(k-1)} + D^{-1}\mathbf{b},$$

and, if $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ converges to $\mathbf{x}$, then

$$\mathbf{x} = D^{-1}(L+U)\mathbf{x} + D^{-1}\mathbf{b}$$

Convergence of the Jacobi & Gauss-Seidel Methods

Example

For example, the Jacobi method has

$$\mathbf{x}^{(k)} = D^{-1}(L+U)\mathbf{x}^{(k-1)} + D^{-1}\mathbf{b},$$

and, if $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ converges to $\mathbf{x}$, then

$$\mathbf{x} = D^{-1}(L+U)\mathbf{x} + D^{-1}\mathbf{b}$$

This implies that

$$D\mathbf{x} = (L+U)\mathbf{x} + \mathbf{b} \quad \text{and} \quad (D-L-U)\mathbf{x} = \mathbf{b}$$

Convergence of the Jacobi & Gauss-Seidel Methods

Example

For example, the Jacobi method has

$$\mathbf{x}^{(k)} = D^{-1}(L+U)\mathbf{x}^{(k-1)} + D^{-1}\mathbf{b},$$

and, if $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ converges to $\mathbf{x}$, then

$$\mathbf{x} = D^{-1}(L+U)\mathbf{x} + D^{-1}\mathbf{b}$$

This implies that

$$D\mathbf{x} = (L+U)\mathbf{x} + \mathbf{b} \quad \text{and} \quad (D-L-U)\mathbf{x} = \mathbf{b}$$

Since $D-L-U = A$, the solution $\mathbf{x}$ satisfies $A\mathbf{x} = \mathbf{b}$.

Convergence of the Jacobi & Gauss-Seidel Methods

The following are easily verified sufficiency conditions for convergence of the Jacobi and Gauss-Seidel methods.

Convergence of the Jacobi & Gauss-Seidel Methods

The following are easily verified sufficiency conditions for convergence of the Jacobi and Gauss-Seidel methods.

Theorem

If A is strictly diagonally dominant, then for any choice of $\mathbf{x}^{(0)}$, both the Jacobi and Gauss-Seidel methods give sequences $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ that converge to the unique solution of $A\mathbf{x} = \mathbf{b}$.

Convergence of the Jacobi & Gauss-Seidel Methods

Is Gauss-Seidel better than Jacobi?

Convergence of the Jacobi & Gauss-Seidel Methods

Is Gauss-Seidel better than Jacobi?

- No general results exist to tell which of the two techniques, Jacobi or Gauss-Seidel, will be most successful for an arbitrary linear system.

Convergence of the Jacobi & Gauss-Seidel Methods

Is Gauss-Seidel better than Jacobi?

- No general results exist to tell which of the two techniques, Jacobi or Gauss-Seidel, will be most successful for an arbitrary linear system.
- In special cases, however, the answer is known, as is demonstrated in the following theorem.

Convergence of the Jacobi & Gauss-Seidel Methods

(Stein-Rosenberg) Theorem

If $a_{ij} \leq 0$, for each $i \neq j$ and $a_{ii} > 0$, for each $i = 1, 2, \ldots, n$, then one and only one of the following statements holds:

(i) $0 \leq \rho(T_g) < \rho(T_j) < 1$

(ii) $1 < \rho(T_j) < \rho(T_g)$

(iii) $\rho(T_j) = \rho(T_g) = 0$

(iv) $\rho(T_j) = \rho(T_g) = 1$

Convergence of the Jacobi & Gauss-Seidel Methods

(Stein-Rosenberg) Theorem

If $a_{ij} \leq 0$, for each $i \neq j$ and $a_{ii} > 0$, for each $i = 1, 2, \ldots, n$, then one and only one of the following statements holds:

(i) $0 \leq \rho(T_g) < \rho(T_j) < 1$

(ii) $1 < \rho(T_j) < \rho(T_g)$

(iii) $\rho(T_j) = \rho(T_g) = 0$

(iv) $\rho(T_j) = \rho(T_g) = 1$

For the proof of this result, see pp. 120–127. of

Young, D. M., Iterative solution of large linear systems, Academic Press, New York, 1971, 570 pp.

Convergence of the Jacobi & Gauss-Seidel Methods

Two Comments on the Thoerem

- For the special case described in the theorem, we see from part (i), namely

$$0 \leq \rho(T_g) < \rho(T_j) < 1$$

Convergence of the Jacobi & Gauss-Seidel Methods

Two Comments on the Thoerem

- For the special case described in the theorem, we see from part (i), namely

$$0 \leq \rho(T_g) < \rho(T_j) < 1$$

that when one method gives convergence, then both give convergence,

Convergence of the Jacobi & Gauss-Seidel Methods

Two Comments on the Thoerem

- For the special case described in the theorem, we see from part (i), namely

$$0 \leq \rho(T_g) < \rho(T_j) < 1$$

that when one method gives convergence, then both give convergence, and the Gauss-Seidel method converges faster than the Jacobi method.

Convergence of the Jacobi & Gauss-Seidel Methods

Two Comments on the Thoerem

- For the special case described in the theorem, we see from part (i), namely

$$0 \leq \rho(T_g) < \rho(T_j) < 1$$

  that when one method gives convergence, then both give convergence, and the Gauss-Seidel method converges faster than the Jacobi method.

- Part (ii), namely

$$1 < \rho(T_j) < \rho(T_g)$$

  indicates that when one method diverges then both diverge, and the divergence is more pronounced for the Gauss-Seidel method.

Questions?

Eigenvalues & Eigenvectors: Convergent Matrices

Theorem

The following statements are equivalent.

(i) A is a convergent matrix.

(ii) $\lim_{n\to\infty} \|A^n\| = 0$, for some natural norm.

(iii) $\lim_{n\to\infty} \|A^n\| = 0$, for all natural norms.

(iv) $\rho(A) < 1$.

(v) $\lim_{n\to\infty} A^n\mathbf{x} = \mathbf{0}$, for every $\mathbf{x}$.

The proof of this theorem can be found on p. 14 of Issacson, E. and H. B. Keller, Analysis of Numerical Methods, John Wiley & Sons, New York, 1966, 541 pp.

Return to General Iteration Methods — Introduction
Return to General Iteration Methods — Lemma
Return to General Iteration Methods — Theorem

Fixed-Point Theorem

Let $g \in C[a, b]$ be such that $g(x) \in [a, b]$, for all x in $[a, b]$. Suppose, in addition, that g' exists on (a, b) and that a constant $0 < k < 1$ exists with

$$|g'(x)| \leq k, \quad \text{for all } x \in (a, b).$$

Then for any number p_0 in $[a, b]$, the sequence defined by

$$p_n = g(p_{n-1}), \qquad n \geq 1$$

converges to the unique fixed point p in $[a, b]$.

Return to the Corrollary to the Fixed-Point Theorem

Functional (Fixed-Point) Iteration

Corrollary to the Fixed-Point Convergence Result

If g satisfies the hypothesis of the Fixed-Point ▸ Theorem then

$$|p_n - p| \leq \frac{k^n}{1-k}|p_1 - p_0|$$

◂ Return to the Corollary to the Convergence Theorem for General Iterative Methods