Eigenvalues and Eigenvectors

Definition:

Let $A \in M_{n\times n}(\mathbb{F})$ where $\mathbb{F}$ may be $\mathbb{R}$ or $\mathbb{C}$

$\vec{x}$ is an **Eigenvector** (EV) of A if $A\vec{x} = \lambda\vec{x}$,

The equation $A\vec{x} = \lambda\vec{x}$ can be interpreted both algebraically and geometrically

where λ is a constant (either $\mathbb{R}$ or $\mathbb{C}$)

and $\vec{x} \in \mathbb{F}^n$, $\vec{x} \neq \vec{0}$

λ is an **Eigenvalue** (ev) of A associated with the Eigenvector (EV) $\vec{x}$

Algebraic Meaning of $A\vec{x} = \lambda\vec{x}$

Note that $A\vec{x} = \lambda\vec{x} \;\Rightarrow\; (A - \lambda I)\vec{x} = \vec{0}$

The (**Eigenvectors** $\{\vec{x}\}$ **+**$\vec{\mathbf{0}}$) form the **Null Space** of the matrix $(A - \lambda I)$ where it should be noted that $\vec{0}$ was explicitly left out from the definition of the eigenvectors.

This subspace of $(A - \lambda I)$ has a special name – **Eigenspace** or **Characteristic Space** of ***A*** **associated with** λ

Example: $A = \begin{pmatrix} 2 & -1 \\ 2 & 4 \end{pmatrix}$ $\quad (A - \lambda I) = \begin{pmatrix} 2-\lambda & -1 \\ 2 & 4-\lambda \end{pmatrix}$

Solving $A\vec{x} = \lambda\vec{x}$ is equivalent to solving the system of linear equations -

$$\begin{pmatrix} 2-\lambda & -1 \\ 2 & 4-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{aligned} (2-\lambda)x_1 - x_2 &= 0 \\ 2x_1 + (4-\lambda)x_2 &= 0 \end{aligned}$$

Since an eigenvector cannot be $\vec{0}$, this system of linear equations can have a non-trivial solution only if $Ker(A - \lambda I) \neq \{\vec{0}\}$

But this is true only if $(A - \lambda I)$ is non-invertible, which can only happen if –

$$det(A - \lambda I) = |A - \lambda I| = 0 \quad \Rightarrow \quad (2-\lambda)(4-\lambda)-(-1)(2)=0$$

Solving this equation, we get Eigenvalues $\lambda_1 = 3+i,\ \lambda_2 = 3-i$

The eigenvectors for each eigenvalue are found by solving -

$$(A - \lambda_1 I)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ and } (A - \lambda_2 I)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Eigenvector for $\lambda_1 = 3+i$: $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = K\begin{pmatrix} \frac{(-1+i)}{2} \\ 1 \end{pmatrix}$

Eigenvector for $\lambda_2 = 3-i$: $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = K\begin{pmatrix} \frac{(-1-i)}{2} \\ 1 \end{pmatrix}$

The Eigenspace is formed by these two vectors along with the zero vector $\vec{0}$

Features of an Invertible Matrix $B \in M_{n\times n}$

1. B is invertible

2. $B\vec{x} = \vec{b}$ has a unique solution $\vec{x}$ $\forall$ $\vec{b} \in \mathbb{R}^n$

3. $rref(B) = \mathbb{I}_n$

4. $rank(B) = n$

5. $im(B) = \mathbb{R}^n$

6. $Ker(B) = \{\vec{0}\}$

All these statements are equivalent

A slight digression –

Question: Why $null(B) = \{0\} \Leftrightarrow B$ is invertible

Answer: The transformation $T: U \rightarrow V$ is invertible if and only if T is *one to one* & *onto*

This implies that - $dim(U) = dim(V)$

Rank Nullity Theorem $\Rightarrow$ $$\boldsymbol{null(T) + rank(T) = dim(U)}$$

$$\{\mathbf{0}\} \qquad\qquad dim(V)$$

Something Interesting and Useful!

- The trace of a matrix (product of its diagonal terms) is equal to the product of its eigenvalues

- The sum of the eigenvalues of a matrix is equal to the determinant of the matrix

Example

What are the eigenvalues and eigenvectors of the $n \times n$ identity matrix $\mathbb{I}_n$?

Is there an eigenbasis for $\mathbb{I}_n$?

Which matrix would diagonalize $\mathbb{I}_n$?

Exercise Problem

Consider $A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 0 \end{pmatrix}$

(a) Find the characteristic polynomial

Ans: $\lambda^3 - 4\lambda^2 + \lambda + 6$

(b) Find the eigenvalues of *A*

Ans: -1, 2, 3

(c) Find the eigenvectors of *A*

Ans: $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

$\lambda = -1$

$(A - \lambda I) = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 1 \end{pmatrix}$ *rref*= $\begin{pmatrix} 1 & 0 & -0.5 \\ 0 & 1 & -0.5 \\ 0 & 0 & 0 \end{pmatrix}$

Geometric Meaning of $A\vec{x} = \lambda\vec{x}$, when λ is real

As indicated earlier, when λ is real, $A\vec{x}$ is parallel to $\vec{x}$.

This implies that a Eigenvector $\vec{x}$, either gets stretched or compressed along its length when acted upon by the transformation matrix *A*

Algebraic Multiplicity & Geometric Multiplicity

Algebraic Multiplicity: Let A be a $N \times N$ matrix and let $\lambda_1, \ldots\ldots \lambda_N$ be the *possibly repeated* eigenvalues of A which solve the characteristic equation

$$det(A - \lambda I) = 0 = (\lambda - \lambda_1) \cdot \cdots \cdot (\lambda - \lambda_N)$$

The eigenvalue λ_n has algebraic multiplicity $\mu(\lambda_n)$ if the characteristic equation has *exactly* $\mu(\lambda_n)$ solutions equal to λ_n

Geometric Multiplicity: Let A be a $N \times N$ matrix and let λ_n be one of the eigenvalues and denote its associated eigenspace by E_n.

The dimension of E_n is referred to as the geometric multiplicity of the eigenvalue λ_n

The Geometric Multiplicity of an eigenvalue is LESS THAN OR EQUAL to its Algebraic Multiplicity

also, **Geometric Multiplicity of eigenvalue λ = nullity$(A - \lambda I)$ = *N* - *rank*$(A - \lambda I)$**

Algebraic Multiplicity & Geometric Multiplicity EXAMPLE

$A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$

Eigenvalues are $\lambda = -1, \lambda = 2$

Eigenvector for λ=-1 is $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Eigenvector for λ=2 is $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Algebraic Multiplicity is 1 for both the eigenvalues

Geometric Multiplicity is 1 for both the eigenvalues as each of the eigenspaces E_{-1} and E_2 is spanned by only one non-zero vector

$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

Eigenvalues are $\lambda = 1$, **TWICE**

Eigenvector for λ=1 is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Algebraic Multiplicity is 2 for the single eigenvalue

Geometric Multiplicity is 1 for this eigenvalue

$(A - \lambda I) = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$ The rank of this matrix is 1.

By the rank-nullity theorem, we get that the nullspace has dimension 1. Hence, the Geometric Multiplicity is 1.

Note that in this case, Geometric Multiplicity ≠ Algebraic Multiplicity.
In the general case, ***Geometric Multiplicity has to be less than or equal to the Algebraic Multiplicity***

IMPORTANT: If for every eigenvalue of ***A***, the geometric and algebraic multiplicities are equal then the matrix ***A*** will be **Diagonizable**

Algebraic Multiplicity & Geometric Multiplicity EXAMPLE

$$A = \begin{pmatrix} 2 & 2 & 2 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{pmatrix}$$

Eigenvalues are $\lambda = 3$ (Algebraic Multiplicity 1)
$\lambda = 2$ (Algebraic Multiplicity 2)

$\lambda = 3$ $\quad (A - 3I) = \begin{pmatrix} -1 & 2 & 2 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ rank = 2 nullity = 1 Eigenvector = $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ **Geometric Multiplicity = 1**

$\lambda = 2$ $\quad (A - 2I) = \begin{pmatrix} 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ rank = 1 nullity = 2 Eigenvectors = $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

Geometric Multiplicity = 2

Since **Geometric Multiplicity = Algebraic Multiplicity** for each of the two roots, the matrix ***A* will be Diagonalizable**

To Diagonalize ***A*** to the Diagonal Matrix ***D***, use the matrix ***S***, so that ***A=SDS***$^{-1}$ and, therefore, ***D=S***$^{-1}$***AS***.

This is discussed subsequently. For this example, we have -

$$S = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 1 \end{pmatrix}$$

Eigenvectors as the columns

$$S^{-1} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & -2 & -2 \\ 0 & -1 & 0 \end{pmatrix}$$

$$D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Eigenvalues on the diagonal

Certain forms of matrices are convenient to work with. For example –

Upper Triangular Form (also for the Lower Triangular Form)

* sum of two upper triangular matrices also upper triangular
* product of two upper triangular matrices also upper triangular
* inverse remains upper triangular
* transpose is lower triangular
* stays upper triangular if multiplied by a scalar
* determinant is the product of the diagonal elements

Diagonal Form

* determinant is the product of the diagonal elements
* inverse of a diagonal matrix is also diagonal with each term being the inverse of the original term
* transpose of the matrix is the same matrix
* multiplication of two diagonal matrices is commutative, i.e. $PQ = QP$
* powers of the matrix are easily computed
* eigenvalues of the matrix are just the diagonal terms of the matrix

Diagonalizable Matrices

$\boldsymbol{A} \rightarrow \boldsymbol{D}$

any n x n matrix → *n x n matrix in diagonal form*

$\boldsymbol{A} \in \boldsymbol{M}_{n\times n}(\mathbb{F})$ is diagonizable over $\mathbb{F}$ if there exists an invertible matrix $\boldsymbol{S}$ over $\mathbb{F}$ such that -

$$A = SDS^{-1}$$

or equivalently, $$D = S^{-1}AS$$ **Similarity Transformation**

Here, $\boldsymbol{S}$ is said to diagonalize $\boldsymbol{A}$

Note that, A and D have the same eigenvalues which are actually the diagonal terms of D

When is a matrix diagonalizable?

A matrix $\boldsymbol{A} \epsilon \boldsymbol{M}_{n \times n}(\mathbb{F})$ is diagonalizable if and only if $\boldsymbol{A}$ has n linearly independent eigenvectors in $\mathbb{F}^n$

A n x n complex matrix that has n distinct eigenvalues is always diagonizable
(n distinct eigenvalues $\Longrightarrow$ n linearly independent eigenvectors)

To find a matrix $\boldsymbol{S}$ which diagonalizes $\boldsymbol{A}$, find a set of linearly independent eigenvectors of $\boldsymbol{A}$.

If there are enough of them, they can be taken to form the columns of the $\boldsymbol{S}$ matrix.

Example Find a matrix that diagonalizes $A = \begin{pmatrix} 2 & -1 \\ 2 & 4 \end{pmatrix}$

Note that this particular matrix was considered earlier

Solve $|A - \lambda I|$=0 to obtain $\lambda_{1,2} = 3 \pm i$ as the eigenvalues of A.

We then use $A\vec{x}_j = \lambda_j \vec{x}_j$, j=1, 2 to obtain the following eigenvectors

$$\vec{x}_1 = \begin{pmatrix} \frac{-1+i}{2} \\ 1 \end{pmatrix}, \vec{x}_2 = \begin{pmatrix} \frac{-1-i}{2} \\ 1 \end{pmatrix} \text{ for } A$$

The column vectors of ***S*** form an eigenbasis for ***A***

Then $S = \begin{pmatrix} \frac{-1+i}{2} & \frac{-1-i}{2} \\ 1 & 1 \end{pmatrix}$ will diagonalize A

$$A = \begin{pmatrix} 2 & -1 \\ 2 & 4 \end{pmatrix} \qquad \lambda_{1,2} = 3 \pm i \qquad \vec{x}_1 = \begin{pmatrix} \frac{-1+i}{2} \\ 1 \end{pmatrix}, \vec{x}_2 = \begin{pmatrix} \frac{-1-i}{2} \\ 1 \end{pmatrix}$$

$$\boldsymbol{S^{-1}AS} = \begin{pmatrix} -i & \frac{1-i}{2} \\ +i & \frac{1+i}{2} \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} \frac{-1+i}{2} & \frac{-1-i}{2} \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3+i & 0 \\ 0 & 3-i \end{pmatrix}$$

$$= \boldsymbol{D}$$

Note that the diagonal terms of ***D*** are the eigenvalues of ***A***

Example $A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$ Characteristic Equation: $(1-\lambda)^2 - 4 = 0$

Eigenvalues are $\lambda_1 = 3, \lambda_2 = -1$

For $\lambda_1 = 3$ $(A - 3I)\overrightarrow{x_1} = \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $rref \left(\begin{array}{cc|c} 1 & -1/2 & 0 \\ 0 & 0 & 0 \end{array}\right)$ $\overrightarrow{x_1} = k\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

For $\lambda_2 = -1$ $(A + I)\overrightarrow{x_2} = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $rref \left(\begin{array}{cc|c} 1 & 1/2 & 0 \\ 0 & 0 & 0 \end{array}\right)$ $\overrightarrow{x_2} = k\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

To diagonalize *A*

$\boldsymbol{D = S^{-1}AS}$

$S = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}$ $S^{-1} = \begin{pmatrix} 1/2 & 1/4 \\ 1/2 & -1/4 \end{pmatrix}$ $D = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$

Example $A = \begin{pmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{pmatrix}$ $Eigenvalues$: $\lambda_1 = 2, \lambda_2 = 1, \lambda_3 = -1$

$\lambda_1 = 2,$ $\begin{pmatrix} -1 & 1 & 2 \\ -1 & 0 & 1 \\ 0 & 1 & -3 \end{pmatrix}\begin{pmatrix} x_{11} \\ x_{12} \\ x_{13} \end{pmatrix} = \vec{0}$ $rref\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ $\overrightarrow{v_2} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$

$\lambda_2 = 1,$ $\begin{pmatrix} 0 & 1 & -2 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{pmatrix}\begin{pmatrix} x_{21} \\ x_{22} \\ x_{23} \end{pmatrix} = \vec{0}$ $rref\left(\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ $\overrightarrow{v_1} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$

$\lambda_3 = -1,$ $\begin{pmatrix} 2 & 1 & -2 \\ -1 & 3 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} x_{31} \\ x_{32} \\ x_{33} \end{pmatrix} = \vec{0}$ $rref\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ $\vec{v}_{-1} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$A = \begin{pmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{pmatrix}$ $\quad$ $Eigenvalues$: $\lambda_1 = 2, \lambda_2 = 1, \lambda_3 = -1$ with the corresponding *Eigenvectors as* $\quad \overrightarrow{v_2} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} \quad \overrightarrow{v_1} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \quad \vec{v}_{-1} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

The matrix A is obviously diagonizable since it has three distinct eigenvalues. The corresponding S and D matrices are -

$$S = \begin{pmatrix} 1 & 3 & 1 \\ 3 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 2 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Its Eigenspace is spanned by the three vectors $\overrightarrow{v_2}$, $\overrightarrow{v_1}$, $\vec{v}_{-1}$

Example

Consider $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

Find its eigenvalues and eigenvectors, and diagonalize it if you can

Eigenvalues are λ =1 (Algebraic Multiplicity 2) and $\lambda = 0$ (Algebraic Multiplicity 1)

For λ =1, $\vec{X}_1 = ker(A - 1 * I) = ker\begin{pmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \text{span}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ Geometric Multiplicity = 1

For λ =0, $\vec{X}_0 = ker(A - 0 * I) = ker\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \text{span}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ Geometric Multiplicity = 1

Since $\vec{X}_1$ and $\vec{X}_0$ span only the $\vec{X}_0$ - $\vec{X}_1$ plane, we are unable to construct an eigenbasis for A. Hence $\boldsymbol{A}$ **is not diagonizable**.
Note also that for λ =1, the Geometric Multiplicity is less than its Algebraic Multiplicity and, therefore, from that too, ***A* is not diagonizable**

Example $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$ Find its eigenvalues and eigenvectors, and diagonalize it if you can

$Characteristic\ Equation$: $(1-\lambda)(3-\lambda) - 8 = 0$

Eigenvalues are $\lambda = 5$ and $\lambda = -1$

For $\lambda = 5$, $\vec{X}_1 = ker(A - 5 * I) = ker\begin{pmatrix} -4 & 2 \\ 4 & -2 \end{pmatrix} = \text{span}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

For $\lambda = -1$, $\vec{X}_2 = ker(A + 1 * I) = ker\begin{pmatrix} 2 & 2 \\ 4 & 4 \end{pmatrix} = \text{span}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Since $\vec{X}_1$ and $\vec{X}_2$ form an eigenbasis for ***A***, ***A*** is diagonizable with -

$$S = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix}$$

Example Consider $A = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$ Find its eigenvalues and eigenvectors, and diagonalize it if you can

Characteristic Equation simplifies to $\lambda(\lambda+3)^2 = 0 \Rightarrow \lambda_1 = 0, \ \lambda_2 = \lambda_3 = -3$

For $\lambda_1 = 0$ $\quad A\overrightarrow{x_1} = \vec{0} \quad rref = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \quad x_{11} = x_{13}, x_{12} = x_{13} \quad \overrightarrow{x_1} = k\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Eigenvector for $\lambda_1=0$

For $\lambda_2 = \lambda_3 = -3$ $\quad (A+3I)\vec{x} = \vec{0} \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\vec{x} = \vec{0} \quad rref = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

So, $\vec{x} = \begin{pmatrix} -r-s \\ r \\ s \end{pmatrix} = r\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ $\quad \overrightarrow{x_2} = k\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \quad \overrightarrow{x_3} = k\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

Eigenvectors for $\lambda_2 = \lambda_3 = 0$

$$A = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$$

$$\lambda_1 = 0 \qquad \overrightarrow{x_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = \lambda_3 = -3 \quad \begin{cases} \overrightarrow{x_2} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \\ \overrightarrow{x_3} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \end{cases}$$

To Diagonalize ***A*** -

$$S = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \qquad S^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

$$\boldsymbol{D = S^{-1}AS} \qquad D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix}$$