Systems of ODEs

eg 1:

$$\frac{dx}{dt} = 2x - xy$$
$$\frac{dy}{dt} = -3y + 0.5xy$$

Coupled ODEs

eg 2:

$$\frac{dx}{dt} = 2x$$
$$\frac{dy}{dt} = -3y$$

decoupled ODEs

Each eq. can be solved separately

$x(t) = c_1 e^{2t}$

$y(t) = c_2 e^{-3t}$

Autonomous? First order ODEs in two variables

$$\frac{dx}{dt} = P(x, y)$$

$$\frac{dy}{dt} = Q(x, y)$$

or

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 2y = \sin y$$

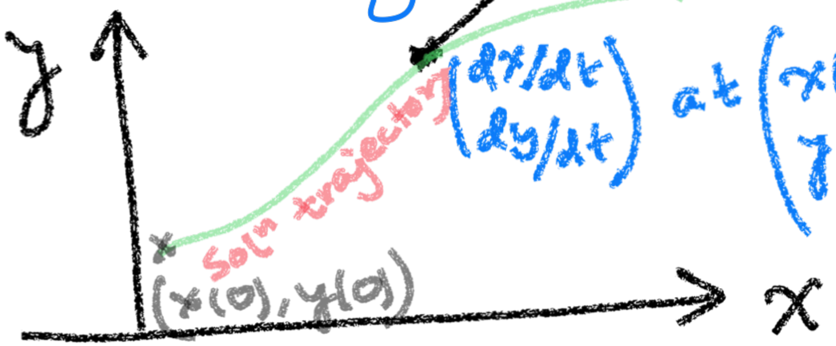
R.H.S. depends explicitly on the dependent variables x, y & only implicitly on the independent variable t.

The solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ represents a parametric curve on the x-y plane

phase plane

Given an initial condition $(x(0), y(0))$; $(x(t), y(t))$ trace out a trajectory (curve) that has the correct tangent vector

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$



$\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix}$ at $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$

Defn

1) Phase plane – xy plane

2) Vector field – collection of tangent vectors defined by the ODE.

3) Trajectory – parametric curve defined by the soln. $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$

4) States – the pts (x, y) of the phase plane

5) phase portrait – collection of trajectories corresponding to various ICs

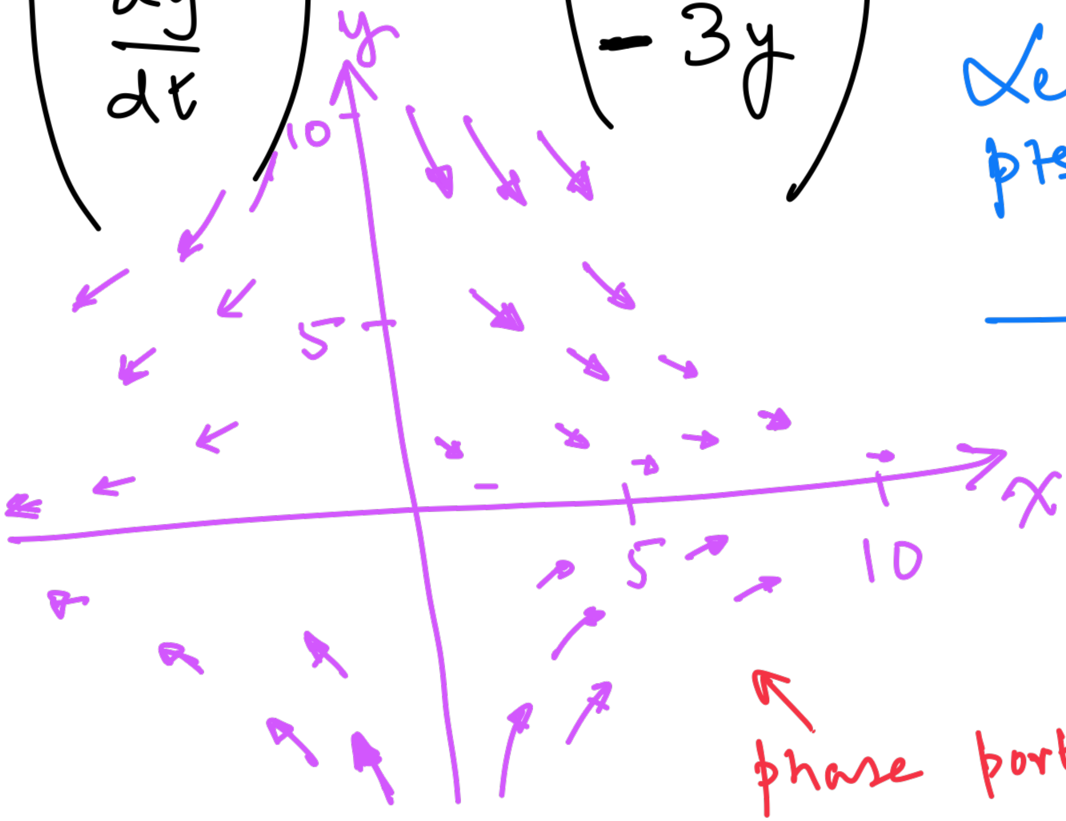
Drawing phase plane trajectories

$$\begin{pmatrix} \dfrac{dx}{dt} \\ \\ \dfrac{dy}{dt} \end{pmatrix} = \begin{pmatrix} 2x \\ \\ -3y \end{pmatrix}$$

$$\frac{dy}{dx} = -\frac{3}{2}\frac{y}{x}$$

Let us take a few test pts. on the phase plane

(x, y)	$\dfrac{dy}{dx}$
$(1, 10)$	-15
$(1, 6)$	-9
$(1, 2)$	-3
$(2, 1)$	-0.75
$(10, 1)$	$-\frac{3}{20} = 0.15$



phase portrait

Equilibrium pts

$$\left.\begin{aligned} \frac{dx}{dt} &= 0 \\ \frac{dy}{dt} &= 0 \end{aligned}\right\}$$

for the above example

$(x, y) = (0, 0)$ is the only eq^m pt.

Types of eq^m. pts

i) Stable $\rightarrow$ attracts nearby sol^ns.

ii) Unstable $\rightarrow$ repels nearby sol^ns.

Q) What type of an eq^m pt. is $(0, 0)$ for the above example ?

Sketching phase plane trajectories

* Matlab : try out the f^n "quiver"
* By hand : use nullclines

→ a v nullcline is an isocline of vertical slopes where $\frac{dx}{dt} = 0$

→ an h nullcline is an isocline of horizontal slopes where $\frac{dy}{dt} = 0$

→ Eq^m pts. occur where a v nullcline intersects an h - nullcline

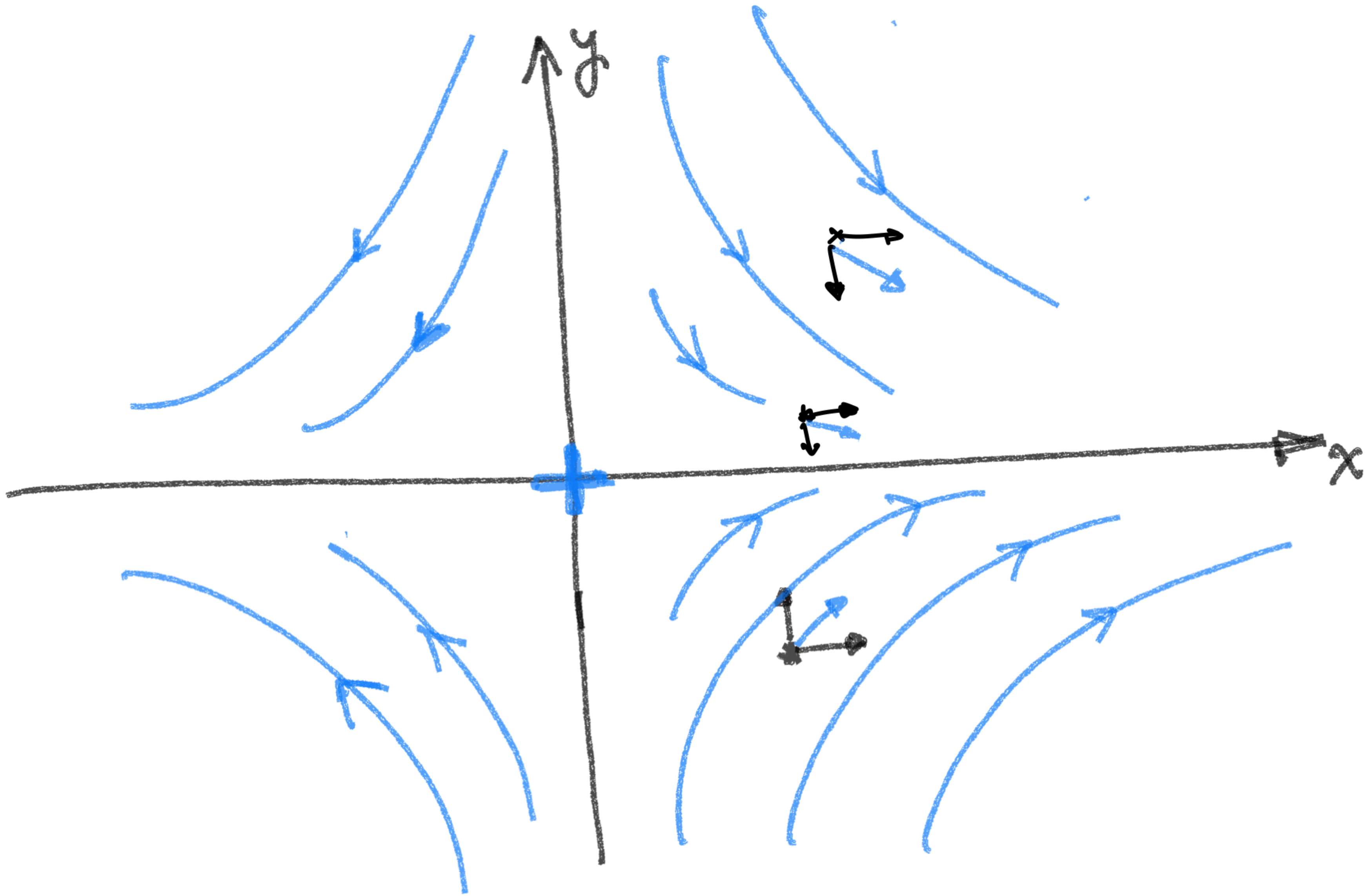
Directions of phase plane trajectories.

	$-ve$	0	$+ve$
$\frac{dx}{dt}$	←	|	→
$\frac{dy}{dt}$	↓	—	↑

* phase plane trajectories do not cross where uniqueness of soln. hold!!

Let's make another attempt at drawing the phase plane trajectories for the system

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} 2x \\ -3y \end{pmatrix}$$



(i) Start w/ the eq^m pt. (0,0)

(ii) Take a few pts. in the phase plane, sketch out the x & h nullcline, then draw the resultant vector