Systems of ODE

We are interested in systems of ODE of the form –

$$\vec{X}' = A(t)\vec{X} + \vec{f}(t) \qquad \text{where} \quad \vec{f}(t) = 0 \qquad \text{Homogenous System}$$

$$\vec{X}(t_0) = \vec{X}_0 \qquad\qquad\qquad\qquad \neq 0 \qquad \text{Non-homogenous System}$$

with solutions of the form - $\vec{X}(t) = \vec{X}_h(t) + \vec{X}_p(t)$

Solution to the homogenous part of the ODE, i.e. with $\vec{f}(t) = 0$

Any particular solution to the linear ODE

Let us begin with the simple case of one ODE, which we will generalize later to the System of ODEs.

(I) Solution by Inspection

Consider the example $y' + 2y = 3$ (Non-homogenous ODE)

The homogenous part of this ODE is $y' + 2y = 0$

with Characteristic Equation $r + 2 = 0$

and Solution $y_h(t) = ce^{-2t}$ c=constant

We now need to find a particular solution $y_p(t)$ to the ODE.

We can see by inspection that $y = \frac{3}{2}$ would be such a solution. (Check!)

Therefore the full solution to the non-homogenous ODE will be $y(t) = y_h(t) + y_p(t)$

$y(t) = ce^{-2t} + \frac{3}{2}$ where c can be found from an initial condition or a known value of $y(t)$ at a given $t = t_1$

It is easy to see that we do have a problem here -

As for the previous example, or for an equation like $y'' + y = t$, the particular solution is easy to guess. (In this case, it is $y_p(t) = t$)

This would be much harder to do in other cases. For example, consider -

$$y'' - y = \sin(t)$$

It turns out that for this, we can use $y_p(t) = -\frac{1}{2}\sin(t)$ but that is not obvious to do

In general, guessing a particular solution to a non-homogenous ODE will be hard to do, which is where the Method of Undetermined Coefficients is useful

However, the Method of Undetermined Coefficients works only for –

(a) Linear ODEs

and (b) certain types of forcing functions, i.e. certain types of $f(t)$

(II) Method of Undetermined Coefficients

For a 2nd order linear ODE

$$ay'' + by' = cy = f(t),$$

the Method of Undetermined Coefficients uses the form of $f(t)$ to predict the form of $y_p(t)$ as per the table shown.

$P_n(t), Q_n(t), A_n(t), B_n(t) \in \mathbb{P}_n$
$A_0, B_0 \in \mathbb{P}_n = \mathbb{R}$
K, ω, C and D are real constants

In (4, 6, 7 & 8), both terms must be included in y_p even if only one term is present in $f(t)$

	$f(t)$	$y_p(t)$
1	K	A_0
2	$P_n(t)$	$A_n(t)$
3	Ce^{Kt}	A_0e^{Kt}
4	$CCos\varpi t + DSin\varpi t$	$A_0Cos\varpi t + B_0Sin\varpi t$
5	$P_n(t)e^{Kt}$	$A_n(t)e^{Kt}$
6	$P_n(t)Cos\varpi t + Q_n(t)Sin\varpi t$	$A_n(t)Cos\varpi t + B_n(t)Sin\varpi t$
7	$Ce^{Kt}Cos\varpi t + DeKtSin\varpi t$	$A_0e^{Kt}Cos\varpi t + B_0e^{Kt}Sin\varpi t$
8	$P_n(t)e^{Kt}Cos\varpi t\ +$ $Q_n(t)e^{Kt}Sin\varpi t$	$A_n(t)e^{Kt}Cos\varpi t\ +$ $B_n(t)e^{Kt}Sin\varpi t$

If any term or terms of y_p are found in y_h (i.e. if such terms are solutions of $ay" + by' + cy = 0$), multiply the expressions of y_p by t (or, if necessary, by t^2) to eliminate the duplication.

Consider the example $y'' + 2y' - 3y = f(t)$

The Homogenous Solution: Solving $y'' + 2y' - 3y = 0$

Characteristic Equation $r^2 + 2r - 3 = 0$

$\Rightarrow \quad r_1 = 1, r_2 = -3$

Therefore $y_h(t) = c_1 e^{t} + c_2 e^{-3t}$

With this form of the solution to the homogenous equation, we can now consider the particular solutions $y_p(t)$ for a few example cases of $f(t)$ next to get the corresponding final solutions $y(t)$.

$f(t) = t^2 + t - 3 \quad \Rightarrow \quad y_p(t) = A_2t^2 + A_1t + A_0$

$f(t) = e^{-t} \quad \Rightarrow \quad y_p(t) = A_0e^{-t}$

$f(t) = te^t \quad \Rightarrow \quad y_p(t) = t(A_1t + A_0)e^t$

Comes because e^t matches e^t in y_h

$f(t) = 2tCos3t + tSin3t \quad \Rightarrow \quad y_p(t) = (A_1t + A_0)Cos3t + (B_1t + B_0)Sin3t$

$f(t) = te^{-2t}Sint \quad \Rightarrow \quad y_p(t) = e^{-2t}\{(A_1t + A_0)Cost + (B_1t + B_0)Sint\}$

Final Solution: $y(t) = c_1e^t + c_2e^{-3t} + y_p(t)$

where the unknown constants may be found if initial conditions are given

Let us consider the ODE $y'' + 2y' - 3y = e^{-t}$ where $f(t) = e^{-t}$

Using $y_h(t)$ obtained earlier and $y_p(t)$ from the previous slide ,

we get – $y(t) = c_1 e^{t} + c_2 e^{-3t} + A_0 e^{-t}$

where $y_p(t) = A_0 e^{-t}$

Since $y_p(t)$ must be a solution of the ODE, we have –

$$A_0 e^{-t} - 2A_0 e^{-t} - 3A_0 e^{-t} = e^{-t} \quad \Rightarrow \quad A_0 = -\frac{1}{4}$$

The remaining constants c_1 and c_2 may be found using the specified initial conditions $y(0)$ and $y'(0)$ or the value of $y(t)$ at two different values of t.

We consider once again a System of ODEs as in the first slide.

For example, suppose we want to solve the following ODE with constant coefficients –

$$y''' + 3y'' + 5y' + 2y = e^{-t}$$

with the initial conditions $y(0) = 1, y'(0) = 3, y''(0) = 2$

Can we turn this into a system of ODEs that look more compact?

To do that, consider making substitutions like the ones given below –

$$x_1 = y \quad \Rightarrow \quad x_1{}' = y' = x_2$$

$$x_2 = y' \quad \Rightarrow \quad x_2{}' = y'' = x_3$$

$$x_3 = y'' \quad \Rightarrow \quad x_3{}' = y''' = -3y'' - 5y' - 2y + e^{-t}$$

This is useful because we can then cast it in the form $\vec{X}' = A\vec{X}(t) + \vec{f}(t)$

where -

$$\vec{X}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} \quad A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -5 & -3 \end{pmatrix} \quad \vec{f}(t) = \begin{pmatrix} 0 \\ 0 \\ e^{-t} \end{pmatrix} \quad \text{and} \quad \vec{X}(0) = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$$

as an Initial Value Problem (IVP)

This will be discussed in a subsequent lecture

………. A Few More Examples …………….

Example: $y'' - 4y' + 4y = te^{2t}$

Characteristic Eq. $r^2 - 4r + 4 = 0$ Double Root at 2 $\Rightarrow$ $y_h(t) = c_1 e^{2t} + c_2 te^{2t}$

The term on the RHS of the ODE indicates we look for $y_P(t)$ of the form $y_P(t) = Ate^{2t} + Be^{2t}$

However, here both terms are linearly dependent with terms in $y_h(t)$, so we instead choose

$$y_P(t) = At^3 e^{2t} + Bt^2 e^{2t}$$

Substituting in the original ODE, we get $y'' - 4y' + 4y = e^{2t}(6At + 2B) = te^{2t}$ $\Rightarrow$ $A = \frac{1}{6}, B = 0$

$$y(t) = c_1 e^{2t} + c_2 te^{2t} + \frac{1}{6} t^3 e^{2t}$$

Example: $y'' + 3y' = \sin t + 2\cos t$

Characteristic Eq. $r^2 + 3r = 0$ Roots at 0, -3 $\Rightarrow$ $y_h(t) = c_1 + c_2 e^{-3t}$

The term on the RHS of the ODE indicates we look for $y_P(t)$ of the form $y_P(t) = A\cos t + B\sin t$

Substituting in the original ODE, we get

$$y'' + 3y' = (-A + 3B)\cos t + (-B - 3A)\sin t = \sin t + 2\cos t$$

Therefore, $A = -\frac{1}{2}$, $B = \frac{1}{2}$

$$y(t) = c_1 + c_2 e^{-3t} + \frac{1}{2}(\sin t - \cos t)$$

Example, Initial Value Problem: $y'' + y' - 2y = 3 - 6t \qquad y(0) = -1, y'(0) = 0$

Characteristic Equation: $r^2 + r - 2 = 0 \Rightarrow (r-1)(r+2) = 0 \Rightarrow r = 1, -2$

Therefore, the solution to the homogenous equation is $y_h(t) = c_1 e^t + c_2 e^{-2t}$

For the particular solution, we can use $y_p(t) = At + B$

Substituting $y_p(t)$ in the original equation, we get $A - 2At - 2B = 3 - 6t \Rightarrow A = 3,\ B = 0$

Therefore, $y(t) = y_h(t) + y_p(t) = c_1 e^t + c_2 e^{-2t} + 3t \qquad y'(t) = c_1 e^t - 2c_2 e^{-2t} + 3$

$y(0) = -1 \Rightarrow c_1 + c_2 = -1, \quad y'(0) = 0 \Rightarrow c_1 - 2c_2 + 3 = 0 \Rightarrow c_1 = -\frac{5}{3},\ c_2 = \frac{2}{3}$

$$y(t) = -\frac{5}{3} e^t + \frac{2}{3} e^{-2t} + 3t$$

Example, Initial Value Problem: $y'' + 4y = t \quad y(0) = 1, y'(0) = -1$

Characteristic Equation: $r^2 + 4 = 0 \quad \Rightarrow \quad r = \pm 2i$

Therefore, the solution to the homogenous equation is $y_h(t) = c_1\cos 2t + c_2\sin 2t$

For the particular solution, we can use $y_p(t) = At + B$

Substituting $y_p(t)$ in the ODE, we get $A = \frac{1}{4}, B = 0 \quad \Rightarrow \quad y_P(t) = \frac{1}{4}t$

Therefore, $y(t) = y_h(t) + y_p(t) = c_1\cos 2t + c_2\sin 2t + \frac{1}{4}t$

$y'(t) = -2c_1\sin 2t + 2c_2\cos 2t + \frac{1}{4}$

$y(0) = 1 \Rightarrow c_1 = 1, \quad y'(0) = -1 \quad \Rightarrow \quad 2c_2 + \frac{1}{4} = -1 \quad \Rightarrow \quad c_1 = 1, \; c_2 = -\frac{5}{8}$

$$y(t) = \cos 2t - \frac{5}{8}\sin 2t + \frac{1}{4}t$$