Newton's method (or Newton - Raphson method) (1)

✱ Most powerful (in terms of rate of conv.) & well known root finding algorithm.

Let $f \in \underbrace{C^2[a,b]}$. Let $p_0 \in [a,b]$ be an approx^n of the root p, $(f(p)=0)$

twice continuously differentiable f^n.

s.t. $f'(p_0) \neq 0$

$|p - p_0| \ll 1$

Taylor expand p about p_0

$$\cancelto{0}{f(p)} = f(p_0) + (p - p_0) f'(p_0) + O((p-p_0)^2)$$

1st order approx^n: $0 = f(p_0) + (p - p_0) f'(p_0)$

$$\Rightarrow p = p_0 - \frac{f(p_0)}{f'(p_0)}$$

So the 1st iterate is $p_1 = p_0 - \frac{f(p_0)}{f'(p_0)}$

n^{th} iterate is $\boxed{p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}}$

$\forall\, n \geq 1.$

Reading Assignment!

* for geometrical interpretation see fig 2.7, pg. 64 of text by Burden & Faires (8th ed^n)

Recall Newton's method iteration:

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}$$

will be problematic when $f'(p_{n-1}) = 0$!

* How to circumvent the above problem?

Define $f'(p_{n-1}) = \lim_{x \to p_{n-1}} \frac{f(x) - f(p_{n-1})}{x - p_{n-1}}$ & let $x = p_{n-2}$

$$\approx \frac{f(p_{n-2}) - f(p_{n-1})}{(p_{n-2} - p_{n-1})}$$

$$\therefore \; p_n = p_{n-1} - \frac{f(p_{n-1})(p_{n-1} - p_{n-2})}{f(p_{n-1}) - f(p_{n-2})} \leftarrow \text{Secant Method}$$

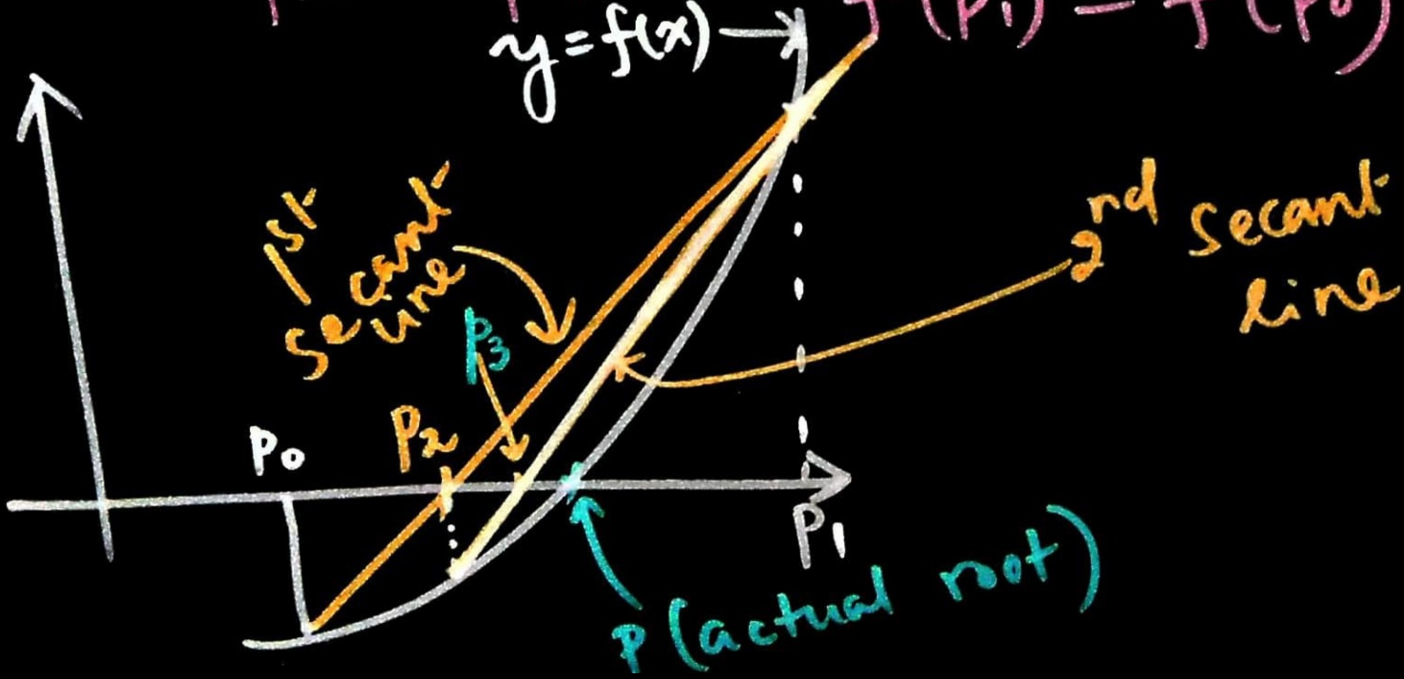
Graphical interpretation of secant method. (4)

$$p_1 = p_0 - \frac{f(p_0)\,(p_0 - p_{-1})}{f(p_0) - f(p_{-1})}$$ starting w/ 2 initial guesses p_0 & p_{-1}

or equivalently,

~~Starting w/ 2 initial guesses p_1 & p_0~~

$$p_2 = p_1 - \frac{f(p_1)\,(p_1 - p_0)}{f(p_1) - f(p_0)}$$



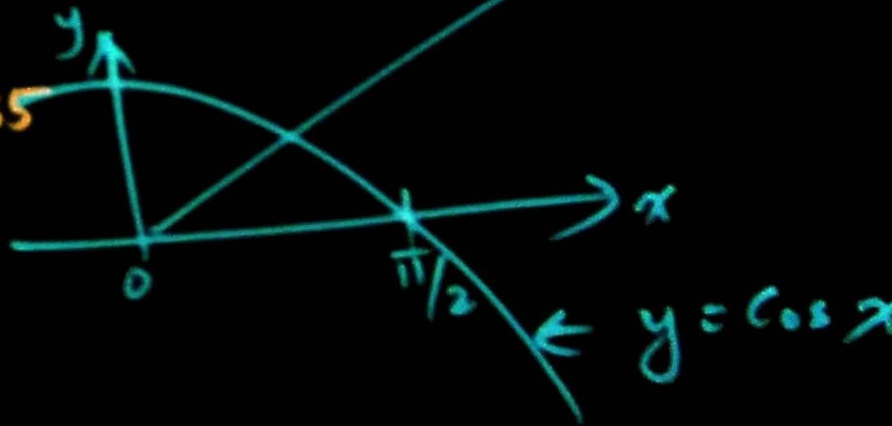
Q) find the soln. to $\cos x = x$.

Ans) We will use the Newton's method

$$f(x) = \cos x - x = 0$$

$$f'(x) = -\sin x - 1$$

$$p_n = p_{n-1} - \frac{\cos(p_{n-1}) - p_{n-1}}{-\sin(p_{n-1}) - 1} \quad ; \quad n \geq 1$$



Take / use

$p_0 = \pi/4 = 0.7853981635$

$p_1 = 0.7395361337$

$p_2 = 0.7390851781$

$p_3 = 0.7390851332$

$p_4 = 0.7390851332$ ← Converges!

Try to repeat the problem w/ $x_{n+1} = g(x_n) = \cos x_n$ & compare speed of convergence.

Th^m (2.5) : Let $f \in C^2[a,b]$.

If $p \in [a,b]$ s.t. $f(p) = 0$ & $f'(p) \neq 0$;

then there exists a $\delta > 0$ s.t.

Newton's method generates a seq.

$\{p_n\}_{n=1}^{\infty} \longrightarrow p$ for any initial guess

$p_0 \in [p-\delta, p+\delta]$

Proof :- Do it yourself

pg. 66 of text by Burden & Faires (8th ed^n.)