Markov Chains: introduction

Discrete Markov Chains (DTMC). (syllabus does not include CTMC).

Markov Chain is a stochastic process where events happen in a sequence s.t. the probability of an event at any given time depends solely on the previous state.

eg. Badminton game.

Mathematically

$\{X_n\}_{n \in I, n \geq 0}$ describes a sequence of events

$$P\left(X_n = x_n \mid X_{n-1} = x_{n-1}, X_{n-2} = x_{n-2}, \ldots, X_0 = x_0\right)$$

$$= P\left(X_n = x_n \mid X_{n-1} = x_{n-1}\right) \Rightarrow \text{Memoryless property.}$$

eg 1 Gambler's ruin

Consider a gambling game in which on any turn you win Rs 1 w/ probability $p = 0.4$ or lose Rs 1 w/ $p' = (1-0.4) = 0.6$. Suppose you adopt a strategy that you quit playing if your fortune reaches Rs 100. Of course if your fortune becomes Rs 0, the casino kicks you out. Design a suitable Markov Model.

Soln:- Let X_n = amt. of money you have after 'n' plays

If $X_n \neq 0$; then $P(X_{n+1} = i+1 \mid X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0)$ $\overset{\text{why?}}{=} P(X_{n+1} = i+1 \mid X_n = i) = 0.4$

State sp. $= \{0, 1, 2, \ldots, 100\}$

So $p_{i,i+1} = 0.4$

$p_{i,i-1} = 0.6$

0 & 100 are absorbing states → $p_{0,0} = 1$ (Casino will kick you out!)

$p_{100,100} = 1$ (You win!)

$$p_{i,j} = \begin{array}{c|c} & j \rightarrow \quad 0 \quad 1 \quad 2 \quad \cdots \quad 99 \quad 100 \\ \hline \begin{array}{c} \downarrow i \\ 0 \\ 1 \\ 2 \\ \vdots \\ 99 \\ 100 \end{array} & \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0.6 & 0 & 0.4 & \cdots & 0 & 0 \\ 0 & 0.6 & 0 & 0.4 \; \cdots & 0 & 0 \\ \vdots & & & & \vdots & \vdots \\ 0 & \cdots & & 0.6 & 0 & 0.4 \\ 0 & \cdots & & 0 & 0 & 1 \end{pmatrix} \end{array}$$

* $p_{i,j} \equiv \mathbb{P}$ in the previous pg. is Known as the Probability Transition Matrix. It is sometimes also Known as the Stochastic matrix.

* DTMC have many applications.

We are working on a Pandemic Mgmt. technology (Epidemiological model) funded by DBT, Govt. of India which is a DTMC model.

Multi-step Transition probabilities.

The probability transition matrix $p_{i,j}$ gives information about going from state i to state j in ONE step.

But we may be interested in knowing about transitions between states in more than one step.

i.e. $p^m_{i,j} \equiv p^m(i,j) = P(X_{n+m} = j \mid X_n = i)$; $m > 1$

i.e. probability of going from state i to state j in $m > 1$ steps.

eg. Social mobility problem: X_n = family's social class in n^{th} generation $= \{1, 2, 3\}$

1 = Low, 2 = Middle, 3 = upper

$$p_{i,j} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0.7 & 0.2 & 0.1 \\ 0.3 & 0.5 & 0.2 \\ 0.2 & 0.4 & 0.4 \end{pmatrix} \end{array}$$

Q1) Your parents were middle class. What is the probability that you are in upper class but your children are lower class?

Soln :- We need to find

$$P(X_2 = 1, X_1 = 3 \mid X_0 = 2) \; ??$$

$X_0 \to$ parents

$X_1 \to$ You

$X_2 \to$ children

Conditional probability
$$\frac{P(X_2 = 1, X_1 = 3, X_0 = 2)}{P(X_0 = 2)}$$

$$= \frac{P(X_2 = 1, X_1 = 3, X_0 = 2)}{P(X_1 = 3, X_0 = 2)} \frac{P(X_1 = 3, X_0 = 2)}{P(X_0 = 2)}$$

Again conditional probability
$$P(X_2 = 1 \mid X_1 = 3, X_0 = 2) \; P(X_1 = 3 \mid X_0 = 2)$$

Markov property
$$P(X_2 = 1 \mid X_1 = 3) \; P(X_1 = 3 \mid X_0 = 2) = p(3,1)\, p(2,3)$$

$$= 0.2 \times 0.2$$

$$= 0.04.$$

Q2) Given you are lower class, what is the probability that your grandchildren are upper class.

Soln:- $p^2(1,3)$ i.e. (1,3) entry in $\mathbb{P}^2$ matrix

$$\mathbb{P}^2 = \begin{pmatrix} 0.7 & 0.2 & 0.1 \\ 0.3 & 0.5 & 0.2 \\ 0.2 & 0.4 & 0.4 \end{pmatrix}\begin{pmatrix} 0.7 & 0.2 & 0.1 \\ 0.3 & 0.5 & 0.2 \\ 0.2 & 0.4 & 0.4 \end{pmatrix} = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & \times & \times & 0.15 \\ 2 & \times & \times & \times \\ 3 & \times & \times & \times \end{array}$$

Chapman Kolmogorov Eqn.

$$p^{m+n}(i,j) = \sum_{K \in S} p^m(i,K)\, p^n(K,j)$$

Why is the above true??

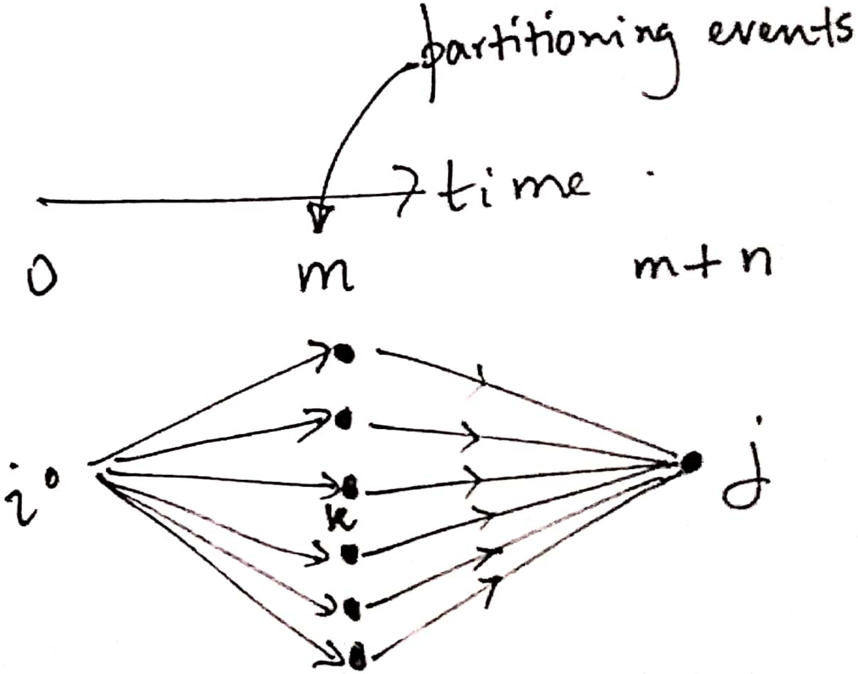
$$p^{m+n}(i,j) = P(X_{n+m} = j \mid X_0 = i)$$

$$= \sum_{k=0}^{\infty} P(X_{m+n} = j, X_m = k \mid X_0 = i)$$

$$= \sum_{k=0}^{\infty} P(X_{m+n} = j \mid X_m = k, X_0 = i)\, P(X_m = k \mid X_0 = i)$$

Law of total probability w/ $X_m = k$ serving as partitioning events

$$\underset{\text{property}}{\overset{\text{Markov}}{=}} \sum_{k=0}^{\infty} p^{n}(k,j)\, p^{m}(i,k)$$



* n^{th} time probability D^n of states
* long time probability D^n of states

We will need :- i) Initial probability D^n of states
ii) probability-transition ~~matrix~~ matrix.

Initial D^n for k-states

$$\vec{\mu}^{(0)} = \left(\mu_1^{(0)}, \mu_2^{(0)}, \dots, \mu_3^{(0)}\right) = \left(P(X_0 = s_1), P(X_0 = s_2), \dots, P(X_0 = s_k)\right)$$

where $\sum_{i=1}^{k} \mu_i^{(0)} = 1$ (b/c axiom of probability)

so n^{th} step D^n of states :- $\vec{\mu}^{(n)} = \vec{\mu}^{(0)} \mathbb{P}^n$

eg (A simple weather model).

Consider a simple model that predicts weather on a given day as follows

→ the weather stays the same on any given day as the previous day 75% of the time

→ 25% of the time it changes

for simplicity, let us consider that there are only 2 states of the weather viz. s_1 = rainy, s_2 = sunny

Q) What is the long time behavior of the weather distribution given that $\vec{\mu}^{(0)} = (1, 0) = (s_1, s_2)$

$$\mathbb{P} = \begin{array}{c c} & \begin{array}{cc} s_1 & s_2 \end{array} \\ \begin{array}{c} s_1 \\ s_2 \end{array} & \begin{pmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{pmatrix} \end{array}$$

$$\vec{\mu}^{(1)} = \vec{\mu}^{(0)}\mathbb{P} = (1 \quad 0)\begin{pmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{pmatrix} = (0.75 \quad 0.25)$$

$$\vec{\mu}^{(2)} = \vec{\mu}^{(0)}\mathbb{P}^2 \left(\text{OR} = \vec{\mu}^{(1)}\mathbb{P}\right) = (1 \quad 0)\begin{pmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{pmatrix}^2 = (0.625 \quad 0.375)$$

$$\vdots$$

$$\vec{\mu}^{(\infty)} = \mu^{(0)}\mathbb{P}^{\infty} = (0.5 \quad 0.5).$$ What is $\vec{\mu}^{(\infty)}\mathbb{P} = ?$

Q) Consider a Markov model of a game of Badminton. For simplicity, let us consider there are only 3 types of shots played by the players viz. {drop, lift, smash} = {D, L, S}

We are interested in formulating/analyzing a winning strategy :-

Shot	Return Shot	w/ probability.
Drop (D)	Drop (D)	1/3
Drop (D)	Lift (L)	1/3
D	S	0
L	D	1/5
L	L	1/5
L	S	2/5

Shot	return shot	w/ probability
S	L	2/5
S	D	1/5
S	S	0

Q1) Identify an appropriate State space.

Q2) Construct a probability transition probability.

Q3) What is the probability of a winning shot given that the final 3 shots in the rally were respectively

X_{n-1}, X_{n-2}, X_{n-3}

(a) smash, lift, lift

(b) smash, drop, lift

(c) drop, lift, smash

Q4) Given a "lifted" serve, what is the probability that there is a "winner" in 3 shots?