Systems of ODE

We are interested in systems of ODE of the form

$$\vec{x}\,' = A(t)\,\vec{x} + \vec{f}(t)$$

$= 0$ (homogeneous)

$\neq 0$ (non-homogeneous)

$$\vec{x}(t_0) = \vec{x}_0$$

Soln:- $\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$

any (particular) sol^n. to the linear ODE.

Sol^n to homogeneous part of the ODE

Let us begin w/ the simple case of one ODE, we will later generalize to the system of ODEs.

I) Soln. by inspection

eg ① $y' + 2y = 3$ (Non-homogeneous ODE)

What is the homogeneous part of this ODE?

$y' + 2y = 0$

Soln:- Ch. eqn. $r + 2 = 0$

$\therefore$ Soln. $y_h(t) = e^{-2t}$

Q) How do we find $y_p(t)$?

By inspection $y = 3/2$ is a soln.

Full soln is $y(t) = y_h(t) + y_p(t)$

$$\boxed{y(t) = e^{-2t} + \frac{3}{2}}$$

(II) <u>Method of undetermined coefficients</u>

* works for linear ODE w/ <u>constant</u> coefficients.
* Certain types of forcing f^n's.

For a 2nd order linear ODE

$$ay'' + by' + cy = f(t);$$

the method of undetermined coeffs. uses the form of $f(t)$ to predict the form of $y_p(t)$ as per the following table!

	$f(t)$	$y_p(t)$
I	K	A_0
II	$P_n(t)$	$A_n(t)$
III	Ce^{Kt}	$A_0 e^{Kt}$
IV	$C\cos\omega t + D\sin\omega t$	$A_0\cos\omega t + B_0\sin\omega t$
V	$P_n(t)e^{Kt}$	$A_n(t)e^{Kt}$
VI	$P_n(t)\cos\omega t + Q_n(t)\sin\omega t$	$A_n(t)\cos\omega t + B_n(t)\sin\omega t$
VII	$Ce^{Kt}\cos\omega t + De^{Kt}\sin\omega t$	$A_0 e^{Kt}\cos\omega t + B_0 e^{Kt}\sin\omega t$
VIII	$P_n(t)e^{Kt}\cos\omega t + \cdots$	$A_n(t)e^{Kt}\cos\omega t + \cdots$

eg. $y'' + 2y' - 3y = f(t)$

We will consider different $f(t)$ as examples. Let us first find $y_h(t)$

Ch. eqn. $r^2 + 2r - 3 = 0 \Rightarrow r_1 = 1, r_2 = -3$

$\therefore y_h(t) = c_1 e^{t} + c_2 e^{-3t}$

Consider a few example cases of $f(t)$ next.

(a) $f(t) = t^2 + t - 3 \Rightarrow y_p(t) = A_2 t^2 + A_1 t + A_0$

(b) $f(t) = e^{-t} \Rightarrow y_p(t) = A_0 e^{-t}$

(c) $f(t) = t e^t \Rightarrow y_p(t) = t(A_1 t + A_0) e^t$

Wait a min!!

Comes b/c e^t matches w/ e^t in y_h (double root)

Comes b/c of

(d) $f(t) = 2t\cos 3t + t\sin 3t \Rightarrow y_p(t) = (A_1 t + A_0)\cos 3t + (B_1 t + B_0)\sin 3t$

(e) $f(t) = te^{-2t}\sin t$

$$\Rightarrow y_p(t) = e^{-2t}\left\{(A_1 t + A_0)\cos t + (B_1 t + B_0)\sin t\right\}$$

Final sol^n :- $y(t) = c_1 e^{t} + c_2 e^{-3t} + y_p(t)$

If ICs are given

$y(0) = 0$; $y'(0) = 0$??

Let us take case (b) $f(t) = e^{-t}$

$$y(t) = c_1 e^{t} + c_2 e^{-3t} + A_0 e^{-t}$$

$y_p' = -A_0 e^{-t}$

$y_p''(t) = A_0 e^{-t}$

$\therefore$ $y_p(t)$ is a soln. to

$$y'' + 2y' - 3y = e^{-t}$$

$$A_0 e^{-t} - 2A_0 e^{-t} - 3A_0 e^{-t} = e^{-t}$$

$$\Rightarrow A_0 = -\frac{1}{4}$$

* the remaining 2 const. c_1 & c_2 can be found using the ICs & $y(t)$ w/ $A_0 = -1/4$

Now, lets return to the system of ODEs in the first slide!

Let us suppose we are asked to solve the ODE w/ constant coeff.

$$\boxed{y''' + 3y'' + 5y' + 2y = e^{-t}}$$

w/ $y(0) = 1$; $y'(0) = 3$; $y''(0) = 2$

Is there a way to turn this to a system of ODE that looks compact?

Consider the substitution

$$\left.\begin{aligned} x_1 &= y \\ x_2 &= y' \\ x_3 &= y'' \end{aligned}\right\} \Rightarrow \begin{aligned} x_1' &= y' = x_2 \\ x_2' &= y'' = x_3 \\ x_3' &= y''' = -3y'' - 5y' - 2y + e^{-t} \end{aligned}$$

Q) Why is this useful?

Ans) B/c this is in $\vec{X}'(t) = \underline{A}\vec{X}(t) + f(t)$ form!

$$\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}; \quad A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -5 & -3 \end{pmatrix}$$

$$\vec{f}(t) = \begin{pmatrix} 0 \\ 0 \\ e^{-t} \end{pmatrix}$$

Initial value problem

And the IVP becomes

$$\vec{x}\,'(t) = A\vec{x}(t) + \vec{f}(t)$$

$$\text{w/} \quad \vec{x}(0) = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$$

How do we solve such systems of ODE?

Coming Soon!!