Tutorial 4: Series and Sequences of Complex Functions, Singularities and Analytic Continuation

Due: April 15 (Monday) in class

1. Determine if the following sequences converge. Do they converge uniformly in the region $0 < \alpha \le |z| \le \beta < \infty$?

   (i) $\{\frac{1}{nz^2}\}_{n=1}^{\infty}$ (ii) $\{\sin(z/n)\}_{n=1}^{\infty}$

2. Compute the integrals: $\lim_{n\to\infty} \int_0^1 nz^{n-1}dz$ and $\int_0^1 \lim_{n\to\infty} nz^{n-1}dz$. Explain your observation.

3. Find the first two non-zero terms of the Laurent expansion of $f(z) = \tan z$ about $z = \pi/2$. What is the radius of convergence of this series.

4. Discuss the analytic continuation of $\log z$ on the complex plane. Is the continuation unique?

5. Discuss the singularity of the function $f(z)$ which is expanded as the following series

$$f(z) = \sum_{n=0}^{\infty} z^{2^n}.$$

6. Expand $f(z) = \frac{z}{(z-2)(z+i)}$ as a Laurent series about $z = 0$ in the following regions:
   (i) $|z| < 1$, (ii) $1 < |z| < 2$, and (iii) $|z| > 2$.

7. Consider the function $F(z) = \int_0^{\infty} e^{3t}e^{-zt}dt$. What is the region of analyticity, A of $F(z)$? Can we extend the region of analyticity of $F(z)$ from A to a bigger region? That is, can we find a region B and a function $\tilde{F}(z)$ such that
   (a) B contains A,
   (b) $\tilde{F}(z)$ is analytic on B, and
   (c) $\tilde{F}(z) = F(z), \ \forall z \in A$.

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