Solving systems of linear ODEs w/ complex eigenvalues

We will develop the theory here for a 2×2 system. The generalization to an $n \times n$ system will follow naturally!

$$\vec{x}' = A_{2\times 2}\vec{x}$$

say has EVs $\lambda_{1,2} = \alpha \pm i\beta$

Corresponding EVs will be $\vec{v}_1, \vec{v}_2 = \vec{p} \pm i\vec{q}$

Complex λs and EVs always appear in complex conjugate pairs!

So the full soln. can be written as

$$\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$$

But since λ_s and $\vec{v}_s$ are complex, we must break up the solution space into real and imaginary parts to investigate the trajectories on the phase plane!

So let's re-write $\vec{x}(t)$ as

$$\vec{x}(t) = \vec{x}_{re}(t) + i\,\vec{x}_{im}(t)$$

How do we do this?

$$\vec{x}(t) = c_1 e^{(\alpha+i\beta)t}(\vec{p}+i\vec{q}) + c_2 e^{(\alpha-i\beta)t}(\vec{p}-i\vec{q})$$

$$= c_1 e^{\alpha t} \underbrace{e^{i\beta t}}_{\substack{\downarrow \text{ Euler's identity} \\ \cos\beta t + i\sin\beta t}} (\vec{p}+i\vec{q}) + c_2 e^{\alpha t} \underbrace{e^{-i\beta t}}_{\substack{\downarrow \\ \cos\beta t - i\sin\beta t}} (\vec{p}-i\vec{q})$$

$$= \underbrace{c_1 e^{\alpha t}(\cos\beta t\,\vec{p} - \sin\beta t\,\vec{q})}_{\vec{x}_{re}(t)} + c_2 i\, \underbrace{e^{\alpha t}(\sin\beta t\,\vec{p} + \cos\beta t\,\vec{q})}_{\vec{x}_{im}(t)}$$

$(c_2 i)$ is a const. b/c $i = \sqrt{-1}$ is a const.

$$\therefore \vec{x}(t) = c_1 \vec{x}_{re}(t) + c_2 \vec{x}_{im}(t)$$

Question :- Are $\vec{x}_{re}(t)$ and $\vec{x}_{im}(t)$ linearly independent solns. ?

<u>Ans</u> :- Lets plug in $\vec{x}(t) = \vec{x}_{re}(t) + i\,\vec{x}_{im}(t)$ in $\vec{x}' = A\vec{x} = A(\vec{x}_{re} + i\,\vec{x}_{im})$

$$\Updownarrow$$

$$\vec{x}' = \vec{x}'_{re} + i\,\vec{x}'_{im} = A(\vec{x}_{re} + i\,\vec{x}_{im})$$

Now comparing real & imaginary parts of L.H.S. and R.H.S. :-

$$\vec{x}'_{re}(t) = A\vec{x}_{re}(t) \quad \text{and} \quad \vec{x}'_{im}(t) = A\vec{x}_{im}(t)$$

Each of real & imag. parts of $\vec{x}(t)$ satisfy the ODE!

And since a 2×2 system $\vec{x}' = A\vec{x}$ has 2 linearly independent soln; $\vec{x}_{re}$ and $\vec{x}_{im}$ suffice!!

Recall the fundamental matrix $X(t) = \begin{pmatrix} | & | \\ \vec{x}_1 & \vec{x}_2 \\ | & | \end{pmatrix}$ is NOT unique!!

Nice thing: $\vec{x}_{re}(t)$ and $\vec{x}_{im}(t)$ can be studied together on the ph-plane!

eg 1) Solve $\vec{x}' = A\vec{x} = \begin{pmatrix} 6 & -1 \\ 5 & 4 \end{pmatrix} \vec{x}$

Soln:- $\begin{pmatrix} 6 & -1 \\ 5 & 4 \end{pmatrix}$ have evs. $\lambda_{1,2} = 5 \pm 2i$

EVs $\vec{v}_{1,2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \pm i \begin{pmatrix} 0 \\ -2 \end{pmatrix}$

$\therefore$ the general soln. is

$$\vec{x}(t) = c_1 \vec{x}_{re}(t) + c_2 \vec{x}_{im}(t)$$

$$= e^{5t} \left\{ c_1 \begin{pmatrix} \cos 2t \\ \cos 2t + 2\sin 2t \end{pmatrix} + c_2 \begin{pmatrix} \sin 2t \\ \sin 2t - 2\cos 2t \end{pmatrix} \right\}$$

where c_1 & c_2 are real constants

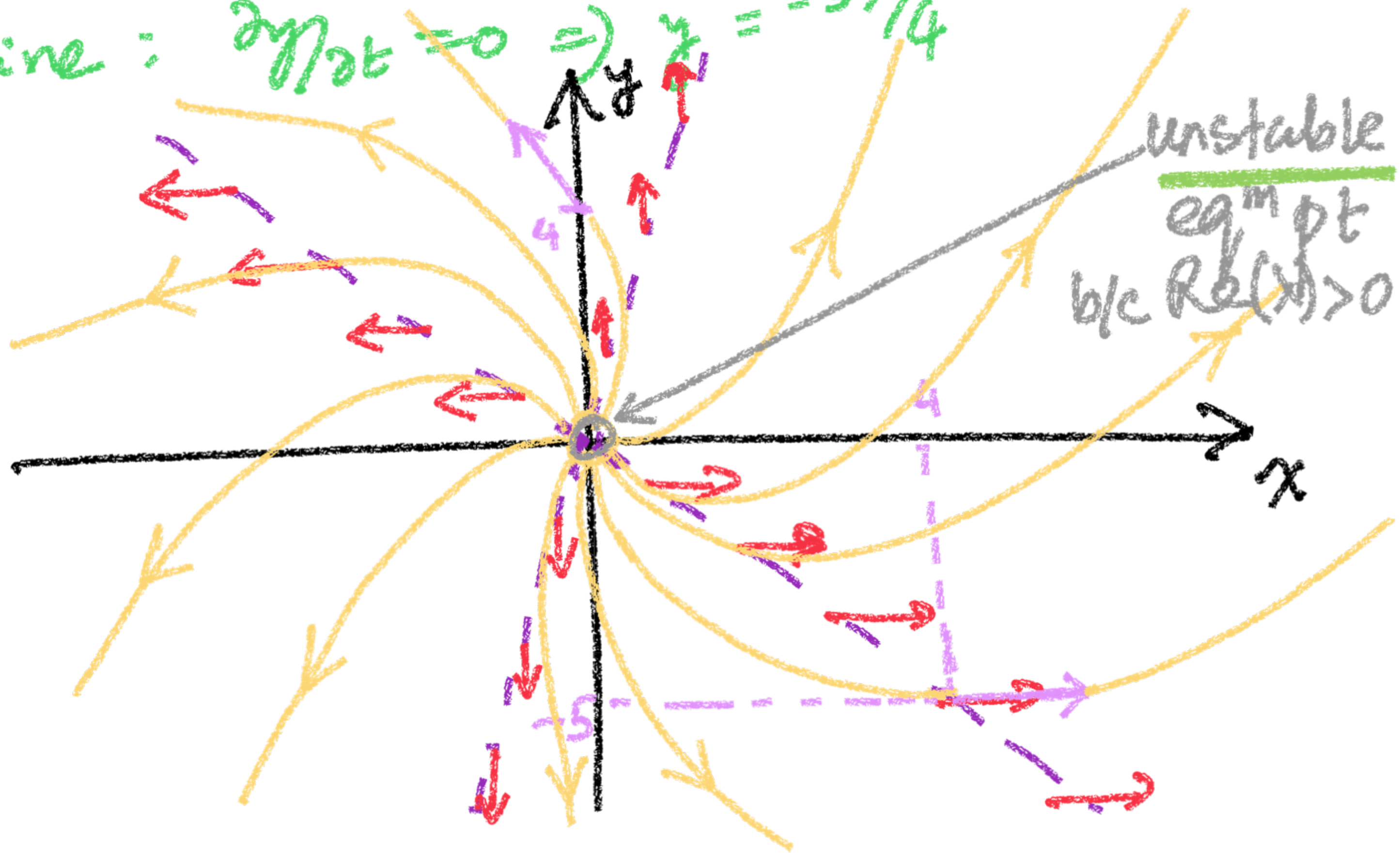
How do we draw the phase portrait ?

$$x' = \begin{pmatrix} \partial x/\partial t \\ \partial y/\partial t \end{pmatrix} = \begin{pmatrix} 6 & -1 \\ 5 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

v - nullcline : $\partial x/\partial t = 0 \Rightarrow y = 6x$

h - nullcline : $\partial y/\partial t = 0 \Rightarrow y = -5x/4$

(x,y)	$\frac{dx}{dt}$	$\frac{dy}{dt}$
$(0,4)$	-4 $\leftarrow$	16 $\uparrow$
$(4,-5)$	29 $\rightarrow$	0



eg 2) $\vec{x}' = A\vec{x}' = \begin{pmatrix} 4 & -5 \\ 5 & -4 \end{pmatrix}\vec{x}$

Soln :- Lets find the evs. of A.

$|A - \lambda I| = 0 \Rightarrow \lambda^2 + 9 = 0 \Rightarrow \lambda_{1,2} = \pm 3i$

EVs :- $\vec{v}_{1,2} = \begin{pmatrix} 5 \\ 4 \mp 3i \end{pmatrix} = \underbrace{\begin{pmatrix} 5 \\ 4 \end{pmatrix}}_{p} \pm i \underbrace{\begin{pmatrix} 0 \\ -3 \end{pmatrix}}_{q}$

$\vec{x}_{re}(t) = \cos 3t \begin{pmatrix} 5 \\ 4 \end{pmatrix} - \sin 3t \begin{pmatrix} 0 \\ -3 \end{pmatrix}$

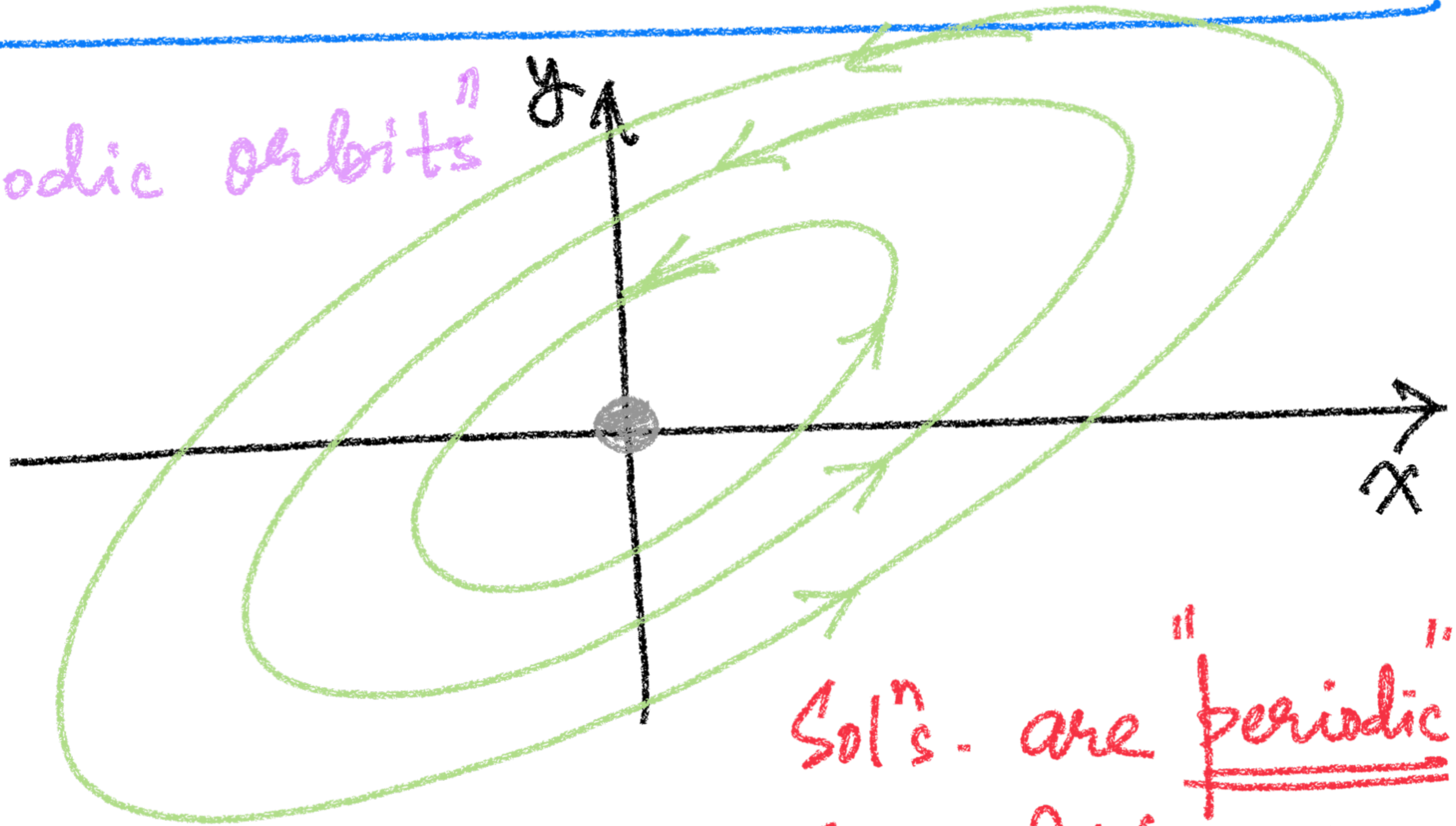
$\vec{x}_{im}(t) = \sin 3t \begin{pmatrix} 5 \\ 4 \end{pmatrix} + \cos 3t \begin{pmatrix} 0 \\ -3 \end{pmatrix}$

General soln. $\vec{x}(t) = c_1 \vec{x}_{re}(t) + c_2 \vec{x}_{im}(t)$

$= c_1 \begin{pmatrix} 5\cos 3t \\ 4\cos 3t + 3\sin 3t \end{pmatrix} + c_2 \begin{pmatrix} 5\sin 3t \\ 4\sin 3t - 3\cos 3t \end{pmatrix}$

How about the phase portrait?

"Periodic orbits"



Sol^{n}s. are "**periodic**"
when EVs are
purely imaginary!