Engineering Maths in Action: EM 103 — Total time: 30 mins
Quiz-2 for section-L2 — December 07, 2023

Full Name: **UID:**

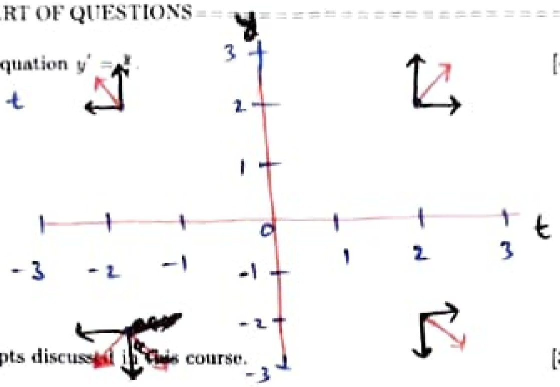
Instructions: You must **not** be in possession of any cheat sheet, notes, or electronic devices like laptops or calculators inside the examination hall. Show **all steps** leading to your final answer to receive any credit for your solution. Merely stating the final answer may not fetch you any credit. The maximum score allotted to each question is mentioned in the square bracket in the right margin. **Maximum score for this quiz is 15.**

==================START OF QUESTIONS==================

1. Draw the direction field for the differential equation $y' = \frac{y}{t}$. [4]

Solution:- Rewriting $\frac{dy}{dx} = y, \frac{dt}{dx} = t$

t	y	dy/dx	dt/dx
2	2	+ve	+ve
-2	2	+ve	-ve
-2	-2	-ve	-ve
2	-2	-ve	+ve



2. Name three real-world applications of concepts discussed in this course. [3]

Solution:-

1. Population dynamics 2. Endemic 3. Page-Rank Algorithm.

3. Find the solution to the system of linear differential equations

$$x' = 2x - y$$
$$y' = 4x + 2y$$

with the initial conditions $x(0) = 1$ and $y(0) = 1$. [4]

Solution:-

$$\frac{dX}{dt} = \begin{bmatrix} 2 & -1 \\ 4 & 2 \end{bmatrix} X \qquad X(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$|A - \lambda I| = 0.$$
$$\begin{vmatrix} 2-\lambda & -1 \\ 4 & 2-\lambda \end{vmatrix} = 0.$$
$$(2-\lambda)^2 + 4 = 0.$$
$$\lambda^2 + 4 - 4\lambda + 4 = 0.$$
$$\lambda^2 - 4\lambda + 8 = 0$$
$$\lambda = 2 \pm 2i$$

for $\lambda = 2+2i$

$$(A - \lambda I)u = 0.$$
$$\begin{bmatrix} -2i & -1 \\ 4 & -2i \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0$$
$$-2iu_1 - u_2 = 0.$$
$$4u_1 - 2iu_2 = 0.$$
$$-2iu_1 = u_2.$$
$$u_1 = i, \quad u_2 = 2 \qquad \begin{bmatrix} i \\ 2 \end{bmatrix}$$

for $\lambda = 2-2i$

$$\begin{bmatrix} 2i & -1 \\ 4 & 2i \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0.$$
$$2iu_1 - u_2 = 0.$$
$$4u_1 + 2iu_2 = 0$$
$$2iu_1 = u_2.$$
$$u_1 = -i, \quad u_2 = 2. \qquad \begin{bmatrix} -i \\ 2 \end{bmatrix}$$

$$X = c_1 e^{(2+2i)t}\begin{bmatrix} i \\ 2 \end{bmatrix} + c_2 e^{(2-2i)t}\begin{bmatrix} -i \\ 2 \end{bmatrix}$$

$$X = e^{2t}\left(c_1(\cos 2t + i\sin 2t)\begin{bmatrix} i \\ 2 \end{bmatrix} + c_2(\cos 2t - i\sin 2t)\begin{bmatrix} -i \\ 2 \end{bmatrix}\right)$$

$$X = e^{2t}\left(c_1\begin{bmatrix} i\cos 2t - \sin 2t \\ 2\cos 2t + 2i\sin 2t \end{bmatrix} + c_2\begin{bmatrix} -i\cos 2t - \sin 2t \\ 2\cos 2t - 2i\sin 2t \end{bmatrix}\right)$$

$$= e^{2t}\left(\begin{bmatrix} -c_1\sin 2t - c_2\sin 2t \\ 2c_1\cos 2t + 2c_2\cos 2t \end{bmatrix} + i\begin{bmatrix} c_1\cos 2t - c_2\cos 2t \\ 2c_1\sin 2t - 2c_2\sin 2t \end{bmatrix}\right)$$

$$= e^{2t}\left((c_1+c_2)\begin{bmatrix} -\sin 2t \\ 2\cos 2t \end{bmatrix} + (c_1-c_2)i\begin{bmatrix} \cos 2t \\ 2\sin 2t \end{bmatrix}\right)$$

$$= e^{2t}\left(c_1\begin{bmatrix} -\sin 2t \\ 2\cos 2t \end{bmatrix} + c_2\begin{bmatrix} \cos 2t \\ 2\sin 2t \end{bmatrix}\right)$$

4. Solve the ODE $y'' - 4y' + 4y = 0$ with the initial conditions $y(0) = 1$ and $y'(0) = 1$. [4]

Solution:-

Auxillary equation

$$r^2 - 4r + 4 = 0.$$

$$(r-2)^2 = 0$$

$$r = 2, 2.$$

$$y(t) = c_1 e^{2t} + c_2 t e^{2t}.$$

$$y(0) = 1$$

$$1 = c_1$$

$$y'(t) = 2c_1 e^{2t} + c_2 e^{2t} + 2c_2 t e^{2t}.$$

$$y'(0) = 1.$$

$$1 = 2c_1 + c_2$$

$$c_2 = 1 - 2 = -1.$$

$$\boxed{y(t) = e^{2t} - t e^{2t}.}$$

$$c_1 \begin{bmatrix} 0 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$c_2 = 1.$$

$$2c_1 = 1$$

$$c_1 = 1/2$$

$$x(t) = e^{2t}\left(\frac{1}{2} \begin{bmatrix} -\sin 2t \\ 2\cos 2t \end{bmatrix} + \begin{bmatrix} \cos 2t \\ 2\sin 2t \end{bmatrix} \right)$$

Hints:

Question 1

- Consider ordered (paired) coordinates (y, t) in all four quadrants.
- Compute and tabulate slopes at each of the above coordinates.
- Draw the slopes as short line segments to populate the slope field.

Question 2

- Write the name of the applications in full.

Question 3

- Write the system of equations in matrix-vector form.
- Compute the eigenvalues and eigenvectors of the coefficient matrix.
- Write the general solution by using the principle of superposition.
- Apply the initial conditions and write the full solution for the given system.

Question 4

- Write the characteristic equation for the given problem.
- Find the roots of the characteristic polynomial.
- Investigate the nature of the above roots and accordingly write the general form of the full solution by applying the principle of superposition.
- Apply the initial conditions and write the full solution for the given differential equation.